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Mathematics

High dimensional cohomology of discrete groups

(Tate cohomology/equivariant cohomology/duality groups/algebraic K-theory)

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ABSTRACT **For a large class of discrete groups Γ, relations are established between the high dimensional cohomology of Γ and the cohomology of the normalizers of the finite subgroups of Γ. The results are stated in terms of a generalization of Tate cohomology recently constructed by F. T. Farrell. As an illustration of these results, it is shown that one can recover a cohomology calculation of Lee and Szczarba, which they used to calculate the odd torsion in $K_3(\mathbf{Z})$.**

F. T. Farrell (private communication, August 1974) has shown that Tate's cohomology theory for finite groups (cf. ref. 1, chap. XII) can be extended to a large class of infinite groups. Farrell's groups $\hat{H}^i(\Gamma,M)$ agree with the ordinary cohomology groups $H^i(\Gamma,M)$ for sufficiently large i, and they are trivial for all coefficient modules M if and only if Γ is torsion-free.

In this note we announce, with some indication of proof, results which make more precise the relation between the Farrell cohomology of Γ and the torsion in Γ. These results are analogous to the results of refs. 2 and 3 on Euler characteristics. They say, roughly speaking, that $\hat{H}^i(\Gamma,M)$ can be computed in terms of the cohomology of the normalizers of the nontrivial finite subgroups of Γ. Moreover, to compute the p-primary component of $\hat{H}^i(\Gamma,M)$ (where p is a prime) one only needs to consider the finite subgroups which are p-groups. For precise statements, see *Theorems 1* and *2* in sections 4 and 5 below; the notation used in those statements is explained in sections 1, 2, and 3. In section 6 we illustrate some of the results by applying them to the group $SL_3(\mathbf{Z})$.

Throughout this paper, Γ will denote a group of virtually finite cohomological dimension, and vcd Γ will denote its virtual cohomological dimension [cf. ref 4, *1.8*].

1. Farrell's cohomology theory

The cohomology groups $\hat{H}^i(\Gamma,M)$ are defined for any integer i and any Γ-module M, and they satisfy:

(*i*) The functors $\hat{H}^i(\Gamma,-)$ form a connected exact sequence of functors on the category of Γ-modules, in the sense of ref. 1, chap. V, §4.

(*ii*) If M is an induced module $\mathbf{Z}\Gamma \otimes_{\mathbf{Z}\Gamma'} M'$, where Γ' is a torsion-free subgroup of finite index and M' is a Γ'-module, then $\hat{H}^i(\Gamma,M) = 0$ for all i.

(*iii*) For $i >$ vcd Γ the functors $\hat{H}^i(\Gamma,-)$ coincide with the ordinary cohomology functors $H^i(\Gamma,-)$ (as a connected sequence of functors).

If Γ is a *virtual duality group* [i.e., the torsion-free subgroups of finite index satisfy Bieri-Eckmann duality (5)], then one also has:

(*iv*) Let D be the Γ-module $H^n(\Gamma,\mathbf{Z}\Gamma)$, where $n =$ vcd Γ. For any Γ-module M there is an exact sequence

$$\cdots \longrightarrow H_{n-i}(\Gamma,D \otimes M) \longrightarrow H^i(\Gamma,M) \longrightarrow \hat{H}^i(\Gamma,M) \longrightarrow H_{n-i-1}(\Gamma,D \otimes M) \longrightarrow \cdots .$$

In particular, $\hat{H}^i(\Gamma,M) \approx H_{n-i-1}(\Gamma,D \otimes M)$ for $i < -1$.

We will often suppress the index i and the coefficient module M from the notation and simply write $\hat{H}(\Gamma)$ instead of $\hat{H}^i(\Gamma,M)$.

The existence of a sequence of functors satisfying properties (*i*), (*ii*), and (*iii*), as well as uniqueness up to canonical isomorphism, can be proved by using the theory of satellites (ref. 1, chap. III). One can also obtain the Farrell cohomology groups directly as the cohomology groups of a cochain complex, as follows:

Let $P = (P_i)_{i \geq 0}$ be a projective resolution of $\mathbf{Z}$ over $\mathbf{Z}\Gamma$. One can show that there exists a chain complex $C = (C_i)_{i \in \mathbf{Z}}$ of projective $\mathbf{Z}\Gamma$-modules, together with a chain map $f: C \rightarrow P$, such that (*a*) C is Γ'-contractible for some torsion-free subgroup Γ' of finite index and (*b*) f is an isomorphism in sufficiently high dimensions. We will call such a complex C [or, more precisely, the triple (C,P,f)] a *complete resolution* for Γ. One verifies easily that if C is a complete resolution then there is a canonical isomorphism $\hat{H}^i(\Gamma,M) \approx H^i[\mathrm{Hom}_\Gamma(C,M)]$.

In the case of a virtual duality group, Farrell constructs a complete resolution by "splicing together" a resolution of $\mathbf{Z}$ and the dual of a finite type resolution of D; this construction leads to property (*iv*).

2. Equivariant cohomology

If Γ operates on a pair (X,X') where X is a finite dimensional semi-simplicial complex and X' is a (possibly empty) subcomplex, then one can define groups $\hat{H}^i_\Gamma(X,X';M)$ which agree with the usual equivariant cohomology groups for i sufficiently large (e.g., $i >$ vcd Γ + dim X). Here M is any Γ-module. In terms of a complete resolution C, these groups are given by

$$\hat{H}^i_\Gamma(X,X';M) = H^i\{\mathrm{Hom}_\Gamma[C,C(X,X';M)]\},$$

where $C(X,X';M)$ is the complex of normalized cochains of (X,X') with coefficients in the underlying abelian group of M, and Γ operates on $C(X,X';M)$ via its action on (X,X') and on M. As before, we will often suppress i and M from the notation. This equivariant cohomology theory has properties analogous to ordinary equivariant cohomology theory. In particular, one can prove analogues of the statements *1.6, 1.7, 1.10,* and *1.11* of ref. 6.

In case Γ is finite, the groups $\hat{H}^i_\Gamma(X,X';M)$ are the same as the groups $J^i(X,X';M)$ studied by Swan (7). As in Swan's work, the main reason for the introduction of these groups is that they enable one to systematically ignore free actions:

PROPOSITION 1. *If Γ acts freely in X-X′ then $\hat{H}_\Gamma(X,X')$ = 0.*

3. Ordered sets

We will be particularly interested in the case where the complex X is the complex $K(S)$ associated to an ordered set S on which Γ operates. In this case, we write $\hat{H}^i_\Gamma(S;M)$ [or $\hat{H}_\Gamma(S)$] instead of $\hat{H}^i_\Gamma[K(S);M]$, and similarly for the relative groups.

For $s \in S$, let $S_s = \{t \in S: t > s\}$ and $\bar{S}_s = \{t \in S: t \geq s\}$. Define the *depth* of s to be the dimension of $K(\bar{S}_s)$, i.e., the largest integer n such that there is a chain $s_0 < \cdots < s_n$ in S with

$s_0 = s$. The isotropy group Γ_s of s operates on the pair $(\bar{S}_s, S_s)$, and we have:

PROPOSITION 2. *There is a spectral sequence,*

$$E_1^{pq} = \prod_{s \in \mathcal{S}_p} \hat{H}_{\Gamma_s}^{p+q} (\bar{S}_s, S_s) \Longrightarrow \hat{H}_{\Gamma}^{p+q} (S),$$

where $\mathcal{S}_p$ is a set of representatives for the Γ-orbits of elements of S of depth p.

Similarly, there is a spectral sequence involving the sets $S^s = \{t: t < s\}$ and $\bar{S}^s = \{t: t \leq s\}$.

4. The finite subgroups of Γ

In this section S will denote the set of nontrivial finite subgroups of Γ. We let Γ act on S by conjugation. Since Γ is virtually torsion-free, $K(S)$ is finite dimensional and $\hat{H}_\Gamma(S)$ is defined. Note that *Proposition 2* relates $\hat{H}_\Gamma(S)$ to the cohomology of the normalizers of the (nontrivial) finite subgroups of Γ.

THEOREM 1. *There is an isomorphism $\hat{H}(\Gamma) \approx \hat{H}_\Gamma(S)$.*

[More precisely, the canonical map $\hat{H}(\Gamma) \to \hat{H}_\Gamma(S)$, induced by the map of $K(S)$ to a point, is an isomorphism.]

Theorem 1 is obtained from the following more general result by taking X to be a point:

PROPOSITION 3. *Let X be a finite dimensional Γ-complex such that the fixed-point subcomplex X^H is acyclic for each $H \in S$. Then $\hat{H}_\Gamma(X) \approx \hat{H}_\Gamma(S)$.*

To prove *Proposition 3* one first replaces X by its cartesian product with a suitable contractible complex, in order to reduce to the case where all the isotropy groups of Γ in X are finite. Letting $X' = \cup_{H \in S} X^H$, one then shows $\hat{H}_\Gamma(X) \approx \hat{H}_\Gamma(X') \approx \hat{H}_\Gamma(S)$.

PROPOSITION 4. *Under the hypotheses of Proposition 3, the canonical map $\hat{H}(\Gamma) \to \hat{H}_\Gamma(X)$ is an isomorphism.*

In fact, *Theorem 1* and *Proposition 3* show that there is an isomorphism $\hat{H}(\Gamma) \approx \hat{H}_\Gamma(X)$, and one can verify that it is given by the canonical map.

5. Localization at p

It is easy to see that the cohomology groups $\hat{H}(\Gamma)$ are torsion-groups, hence we have a primary decomposition

$$\hat{H}(\Gamma) = \bigoplus_p \hat{H}(\Gamma)_{(p)},$$

and similarly for equivariant cohomology.

Let S_p be the set of nontrivial finite p-subgroups of Γ, where p is a fixed prime.

PROPOSITION 5. *Let X be a finite dimensional Γ-complex such that X^H is acyclic for each $H \in S_p$. Then the canonical map $\hat{H}(\Gamma) \to \hat{H}_\Gamma(X)$ induces an isomorphism on p-primary components:*

$$\hat{H}(\Gamma)_{(p)} \approx \hat{H}_\Gamma(X)_{(p)}.$$

This is proved by using a transfer argument (as in ref. 1, chap. XII, §10) to compute $\hat{H}(\Gamma)_{(p)}$ and $\hat{H}_\Gamma(X)_{(p)}$ in terms of the cohomology and equivariant cohomology of subgroups of Γ which have only p-torsion. One then applies *Proposition 4* to these subgroups.

It was shown in ref. 3, §3, that the hypothesis of *Proposition 5* is satisfied for $X = K(S_p)$. Consequently:

THEOREM 2. *The canonical map $\hat{H}(\Gamma) \to \hat{H}_\Gamma(S_p)$ induces an isomorphism*

$$\hat{H}(\Gamma)_{(p)} \approx \hat{H}_\Gamma(S_p)_{(p)}.$$

COROLLARY. *Let p be a prime such that every nontrivial p-subgroup of Γ contains a unique subgroup of order p. Let Φ be a set of representatives for the conjugacy classes of subgroups of Γ of order p, and let $N(P)$ for $P \in \Phi$ be the normalizer of P in Γ. Then the restriction maps $\hat{H}(\Gamma) \to \hat{H}[N(P)]$ induce an isomorphism*

$$\hat{H}(\Gamma)_{(p)} \xrightarrow{\approx} \prod_{P \in \Phi} \hat{H}[N(P)]_{(p)}.$$

6. Example

In this section we give an example where some of the above results are particularly easy to apply. Further examples will be given elsewhere.

Let $\Gamma = SL_3(\mathbf{Z})$. Then Γ has only 2-torsion and 3-torsion, so the same is true of $\hat{H}(\Gamma)$. Up to conjugacy Γ has only two subgroups of order 3, P_1 and P_2, generated by

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

respectively. Their normalizers, N_1 and N_2, are dihedral of orders 6 and 12. The corollary of *Theorem 2* is applicable with $p = 3$, and we obtain:

$$\hat{H}(\Gamma)_{(3)} \approx \hat{H}(N_1)_{(3)} \oplus \hat{H}(N_2)_{(3)} \approx \hat{H}(P_1)^{N_1} \oplus \hat{H}(P_2)^{N_2}.$$

Taking $\mathbf{Z}$ as coefficient module, for example, we find

$$\hat{H}^i(\Gamma,\mathbf{Z})_{(3)} \approx \begin{cases} \mathbf{Z}/3 \oplus \mathbf{Z}/3 & i \equiv 0 \pmod 4 \\ 0 & \text{otherwise.} \end{cases} \qquad [1]$$

It is known from Borel-Serre (8) that Γ is a virtual 3-dimensional duality group with $H^3(\Gamma,\mathbf{Z}\Gamma) \approx \mathrm{St}$, the Steinberg module associated to $SL_3(\mathbf{Q})$. It is also known that $H^1(\Gamma,\mathbf{Z}) = H^2(\Gamma,\mathbf{Z}) = H_0(\Gamma,\mathrm{St}) = 0$ (cf. refs. 9 and 10). Using these known results, together with properties (*iii*) and (*iv*) of §1, we obtain from [1] the following result:

Let $\mathcal{C}$ be the Serre class of finite abelian 2-groups. For $i > 0$ there are $\mathcal{C}$ isomorphisms

$$H^i(\Gamma,\mathbf{Z}) \underset{\mathcal{C}}{\approx} \begin{cases} \mathbf{Z}/3 \oplus \mathbf{Z}/3 & i \equiv 0 \pmod 4 \\ 0 & \text{otherwise.} \end{cases} \qquad [2]$$

$$H_i(\Gamma,\mathrm{St}) \underset{\mathcal{C}}{\approx} \begin{cases} \mathbf{Z}/3 & i = 2 \\ \mathbf{Z} & i = 3 \\ \mathbf{Z}/3 \oplus \mathbf{Z}/3 & i \equiv 2 \pmod 4,\ i > 2 \\ 0 & \text{otherwise.} \end{cases} \qquad [3]$$

In particular, we have recovered the Lee-Szczarba result (10) that $H_1(\Gamma,\mathrm{St}) = 0 \bmod \mathcal{C}$, which is a crucial step in their proof that $K_3(\mathbf{Z}) \underset{\mathcal{C}}{\approx} \mathbf{Z}/3$.

Remark. C. Soulé ["Cohomologie de $SL_3(\mathbf{Z})$," preprint] has independently obtained [2] and has, in addition, calculated the 2-torsion in $H^*(\Gamma,\mathbf{Z})$. Moreover, he has pointed out that his methods can be used to calculate $\hat{H}(\Gamma,\mathbf{Z})$ as well. One can therefore recover [3] by Soulé's methods, and one can also calculate the 2-torsion in $H^*(\Gamma,\mathrm{St})$.

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