Proceedings of the International Congress of Mathematicians
Helsinki, 1978

Cohomology of Infinite Groups

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This is a survey of recent results in the cohomology theory of infinite groups, with emphasis on the theory of groups of finite virtual cohomological dimension. (Recall from **[24]** that if Γ is a group which has torsion-free subgroups of finite index, then all such subgroups have the same cohomological dimension; this common dimension is called the *virtual cohomological dimension* of Γ and denoted vcd Γ.)

1. Euler characteristics. 1.1. If Γ is a group such that $H_i(\Gamma, \boldsymbol{Q})$ is finite dimensional over $\boldsymbol{Q}$ for all i and is trivial for all but finitely many i, then we set $\tilde{\chi}(\Gamma) = \sum_i (-1)^i \dim H_i(\Gamma, \boldsymbol{Q})$. We will say that a group Γ has *finite homological type* if (i) vcd $\Gamma < \infty$ and (ii) $H_*(\Gamma', \boldsymbol{Z})$ is finitely generated for every torsion-free subgroup Γ' of finite index. We then define the *Euler characteristic* $\chi(\Gamma) \in \boldsymbol{Q}$ by $\chi(\Gamma) = \tilde{\chi}(\Gamma')/(\Gamma : \Gamma')$, where Γ' is any such subgroup; it is shown in **[10]** that this is independent of the choice of Γ'. It agrees with the Euler characteristic studied by Wall **[39]** and Serre **[24]** if Γ is of "type (VFL)".

1.2. It is immediate from the definition that $d \cdot \chi(\Gamma) \in \boldsymbol{Z}$, where d is the greatest common divisor of the indices of the torsion-free subgroups Γ' of finite index. But one can, in fact, prove the sharper result that $m \cdot \chi(\Gamma) \in \boldsymbol{Z}$, where m is the least common multiple of the orders of the finite subgroups of Γ (cf. **[10]** or **[13]**). In addition, there are a number of formulas which yield more precise information about $\chi(\Gamma)$ in terms of the torsion in Γ. For example, let Ψ be a set of representatives for the conjugacy classes of elements of Γ of finite order, and assume for

* Partially supported by a grant from the National Science Foundation of the United States.

each $s\in\Psi$ that the centralizer $Z(s)$ is of finite homological type. Then one can prove that Ψ is finite and that

$$(*)\qquad \tilde{\chi}(\Gamma)=\sum_{s\in\Psi}\chi(Z(s)).$$

(More generally, if $\Gamma'\subseteq\Gamma$ is an arbitrary normal subgroup of finite index, then there is a Lefschetz number formula for the action of Γ/Γ' on $H_*(\Gamma', \boldsymbol{Q})$, cf. [14, § 6]; $(*)$ is the special case $\Gamma'=\Gamma$.) In particular, since $\tilde{\chi}(\Gamma)\in\boldsymbol{Z}$, we obtain $\chi(\Gamma)\equiv-\sum_{s\in\Psi'}\chi(Z(s)) \bmod \boldsymbol{Z}$, where $\Psi'=\Psi-\{1\}$; this can be regarded as a formula for the "fractional part" of $\chi(\Gamma)$ in terms of the torsion in Γ.

There is also a formula for the "p-fractional part" of $\chi(\Gamma)$, where p is a prime ([11], [23]; see also [13]): Let $\mathscr{A}_p$ be the set of nontrivial elementary abelian p-subgroups of Γ. [An elementary abelian p-group is a group isomorphic to $(\boldsymbol{Z}_p)^r$ for some $r<\infty$, where $\boldsymbol{Z}_p=\boldsymbol{Z}/p\boldsymbol{Z}$.] If the normalizer $N(A)$ has finite homological type for each $A\in\mathscr{A}_p$, then $\chi(\Gamma)\equiv\chi_\Gamma(\mathscr{A}_p) \bmod \boldsymbol{Z}_{(p)}$, where $\boldsymbol{Z}_{(p)}$ denotes $\boldsymbol{Z}$ localized at p and $\chi_\Gamma(\mathscr{A}_p)$ is an "equivariant Euler characteristic". Moreover, one can show that the latter is given by

$$\chi_\Gamma(\mathscr{A}_p)=\sum_{r\geqq 1}(-1)^{r-1}p^{r(r-1)/2}\sum_{A\in\mathscr{A}_p^r}\chi(N(A)),$$

where $\mathscr{A}_p^r$ is a set of representatives for the conjugacy classes of elementary abelian p-subgroups of Γ of rank r. [Our hypothesis implies that there are only finitely many such conjugacy classes.]

The results described above have applications to group theory and number theory ([10], [11]), as well as to the study of the finite subgroups of the exceptional Chevalley groups over $\boldsymbol{Z}$ [26].

1.3. Suppose now that Γ is a group such that $\boldsymbol{Q}$, regarded as a module over the group algebra $\boldsymbol{Q}\Gamma$, admits a projective resolution of finite length, $0\to P_n\to\cdots\to P_0\to\boldsymbol{Q}\to 0$, with each P_i finitely generated. (Γ is then said to be of type (FP) over $\boldsymbol{Q}$.) We then set (cf. [33]) $E(\Gamma)=\sum(-1)^i r(P_i)$, where $r(\ )$ denotes the Hattori–Stallings rank. This "complete Euler characteristic" is a $\boldsymbol{Q}$-linear combination of Γ-conjugacy classes. We denote by $e(\Gamma)$ the coefficient of the conjugacy class of 1; this is the Euler characteristic of Γ in the sense of [3], [15], and [34]. Like the Euler characteristic χ defined in 1.1 above, e agrees with the Wall–Serre Euler characteristic if Γ is of type (VFL). It is not known whether $e(\Gamma)=\chi(\Gamma)$ whenever both are defined, but this is easily seen to be true if Γ is residually finite [3]; more generally, they are equal if Γ has a subgroup Γ' of finite index such that $E(\Gamma')$ is concentrated at the conjugacy class of 1. A related question is whether $e(\Gamma)=\tilde{\chi}(\Gamma)$ whenever Γ is torsion-free and of type (FP) over $\boldsymbol{Q}$. This is known to be true by results of Bass [3] if Γ satisfies a certain "condition D", which holds for instance if Γ is a linear group.

1.4. Bass's results imply further that $E(\Gamma)$ is supported on the conjugacy classes of elements of finite order if Γ is of type (FP) over $\boldsymbol{Q}$ and satisfies condition D.

Additional results about $E(\Gamma)$ can be obtained by using the methods of **[10]**. One can prove, for example (cf. **[14]**), under suitable hypotheses on Γ, the following formula suggested by Serre:

$$(**) \qquad E(\Gamma) = \sum_{s \in \Psi} e(Z(s)) \cdot [s],$$

where Ψ is as in 1.2 and $[s]$ is the conjugacy class of s. This should be thought of as a refinement of the formula $(*)$ above. Indeed, if $(**)$ holds then one easily deduces $(*)$, but with χ replaced by e.

The hypotheses on Γ under which $(**)$ has been proved are quite complicated, but we can describe a large family $\mathscr{F}$ of examples for which $(**)$ has been proved, as follows. Let $\mathscr{F}_0$ be the class of finite groups; assuming $\mathscr{F}_{n-1}$ has been defined, let $\mathscr{F}_n$ be the class of groups Γ which admit a simplicial action on a complex X such that (i) X/Γ is compact, (ii) the isotropy group Γ_σ is in $\mathscr{F}_{n-1}$ for each simplex σ of X, and (iii) the fixed-point set X^s is contractible for each $s \in \Gamma$ of finite order. Then $\mathscr{F}_0 \subset \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \ldots$, and we set $\mathscr{F} = \bigcup \mathscr{F}_n$. The family $\mathscr{F}$ includes all arithmetic groups (which are in $\mathscr{F}_1$ as a consequence of [7]), as well as the S-arithmetic groups in the reductive case (these are in $\mathscr{F}_2$, cf. **[8**, § 6]. I do not know an algebraic characterization of $\mathscr{F}$, nor do I know any examples of groups of type (FP) over $\boldsymbol{Q}$ which are not in $\mathscr{F}$.

2. Farrell cohomology. F. T. Farrell **[17]** has shown that the Tate cohomology theory for finite groups can be extended to the class of groups Γ such that $\operatorname{vcd} \Gamma < \infty$. Farrell's theory yields cohomology groups $\hat{H}^i(\Gamma)$ $(i \in \boldsymbol{Z})$, such that $\hat{H}^i = H^i$ for $i > \operatorname{vcd} \Gamma$. If Γ is a "virtual duality group", then one can describe $\hat{H}^i$ for $i < -1$ as a homology functor $\tilde{H}_{n-i-1} = H_{n-i-1}(\Gamma, D \otimes_Z -)$, where $n = \operatorname{vcd} \Gamma$ and D is the Γ-module $H^n(\Gamma, \boldsymbol{Z}\Gamma)$; moreover, there is an exact sequence relating $\{\hat{H}^i\}_{-1 \leq i \leq n}$, $\{H^i\}_{0 \leq i \leq n}$, and $\{\tilde{H}_i\}_{0 \leq i \leq n}$ (cf. [17], [13]). This exact sequence generalizes the sequence $0 \to \hat{H}^{-1} \to H_0 \xrightarrow{N} H^0 \to \hat{H}^0 \to 0$ which one has if Γ is finite, where N is the "norm map". (*Note:* If Γ is finite then $n=0$ and $D = \boldsymbol{Z}$, with trivial Γ-action.) The Farrell cohomology groups are all torsion groups. In fact, if d and m are the integers defined in 1.2, then $d \cdot \hat{H}^*(\Gamma) = 0$, but it is not known whether one always has $m \cdot \hat{H}^*(\Gamma) = 0$.

It is shown in **[12]** and **[13]** that a great deal of information about $\hat{H}^*(\Gamma)$ (and hence about $H^i(\Gamma)$ for $i > \operatorname{vcd} \Gamma$) can be extracted from the finite subgroups of Γ. For example, $\hat{H}^*(\Gamma)$ is periodic if and only if every finite subgroup of Γ has periodic cohomology in the usual sense. (This improves a result of Venkov **[36]**.) Similarly, if p is a prime then the p-primary component $\hat{H}^*(\Gamma)_{(p)}$ is periodic if and only if $\hat{H}^*(G)_{(p)}$ is periodic for every finite subgroup $G \subseteq \Gamma$, i.e., if and only if Γ contains no subgroups isomorphic to $\boldsymbol{Z}_p \times \boldsymbol{Z}_p$. Another result, analogous to that described in 1.2 on the p-fractional part of the Euler characteristic, is that $\hat{H}^*(\Gamma)_{(p)} \approx \hat{H}^*_\Gamma(\mathscr{A}_p)_{(p)}$, the latter being "equivariant Farrell cohomology". If Γ contains no subgroups isomorphic to $\boldsymbol{Z}_p \times \boldsymbol{Z}_p$ (i.e., if $\hat{H}^*(\Gamma)_{(p)}$ is periodic), this isomorphism takes the simple form $\hat{H}^*(\Gamma)_{(p)} \approx \prod_{P \in \mathscr{P}} \hat{H}^*(N(P))_{(p)}$, where $\mathscr{P}$ is a set of representatives for the conjugacy classes of subgroups of order p. See **[22]** for earlier results relating the cohomology of Γ to the elementary abelian p-subgroups.

3. Cohomology calculations. The proofs of the results described in §2 are based on the fact, due to Serre [**24**, 1.7], that if vcd $\Gamma < \infty$ then there exists a contractible finite-dimensional space X on which Γ acts properly (and hence with finite isotropr, groups). The arguments are of a general nature. For a given group Γ, howeveyl one can often get more precise information about $H^*(\Gamma)$ by choosing X convenienty and making a more detailed analysis.

Consider, for example, the case $\Gamma = SL_n(\boldsymbol{Z})$. Classically one takes X to be the symmetric space $\mathrm{SL}_n(\boldsymbol{R})/SO_n(\boldsymbol{R})$, which can be identified with the space of positive definite real quadratic forms in n variables, modulo multiplication by positive scalars. This choice of X, however, is inconvenient for calculation because $\Gamma \backslash X$ is noncompact. One way to remedy this is to replace X by its Borel–Serre "bordification" $\bar{X}$ [7]. This was done, for example, by Lee and Szczarba **[20]**, who were thereby able to completely compute the integral cohomology of the principal congruence subgroup of level 3 of $SL_3(\boldsymbol{Z})$. The space $\bar{X}$ was also used by Lee **[19]** in his construction of several families of "unstable" elements of $H^*(SL_n(\boldsymbol{Z}), \boldsymbol{R})$, i.e., cohomology classes which do not come from $H^*(SL(\boldsymbol{Z}), \boldsymbol{R})$. (Recall that the latter was computed by Borel **[5]**; it is an exterior algebra with one generator of degree $4i+1$ for each integer $i \geqslant 1$.)

A different approach is to replace X by a contractible $SL_n(\boldsymbol{Z})$-invariant subspace X' with compact quotient $SL_n(\boldsymbol{Z}) \backslash X'$. Soulé ([**27**], [**31**]) and Ash ([**1**], [**2**]; see also [**13**, § 2, Ex. 5]) have shown that there always exists such an X' of dimension $n(n-1)/2$; this had previously been observed by Serre **[25]** in the case $n=2$. (We remark that vcd $SL_n(\boldsymbol{Z}) = n(n-1)/2$, so X' has the smallest possible dimension for a contractible space on which $SL_n(\boldsymbol{Z})$ acts properly.) The most striking result obtained in this way is the complete calculation by Soulé **[27]** of $H^*(SL_3(\boldsymbol{Z}), \boldsymbol{Z})$. This was achieved by using an explicit cell-decomposition of X' in order to compute the spectral sequence of equivariant cohomology theory (cf. **[18]** or **[22]**)

$$E_2^{pq} = H^p(\Gamma \backslash X', \mathscr{H}_\Gamma^q) \Rightarrow H^{p+q}(\Gamma).$$

(Here $\mathscr{H}_\Gamma^q$ is a certain sheaf on $\Gamma \backslash X'$ whose stalks are the groups $H^q(\Gamma_x)$, where $x \in X'$ and Γ_x is the isotropy group of x.)

Still a third method was used by Lee and Szczarba **[21]** to partially compute $H^*(SL_n(\boldsymbol{Z}))$ for $n=4$ and 5. They replaced X by an enlargement X^* due to Voronoi **[37]**, which comes equipped with a cell-decomposition compatible with the $SL_n(\boldsymbol{Z})$-action. Their calculations were pushed further by Soulé ([**29**], [**31**]). Similar methods have been applied in **[32]** to the group $SL_3(\boldsymbol{Z}[\sqrt{-1}])$.

Further information on the cohomology of $SL_n(\boldsymbol{Z})$ and other arithmetic groups has been obtained by Eckmann [private communication] and Soulé ([**28**], [**30**], [**31**];

see also **[16]**, **[35]**) by studying characteristic classes. In particular, many interesting examples of torsion classes in $H^*(SL_n(\mathbf{Z}), \mathbf{Z})$ have been obtained in this way.

4. Further results. I have, of course, had to omit many topics from this survey. In particular, I would like to call attention to: (a) the work of Bieri and others on cohomological dimension, duality groups, and related matters (see **[4]** and the references cited there); (b) stability theorems of Quillen (unpublished), Wagoner **[38]**, and R. Charney [unpublished] for $H_*(GL_n(R))$ for suitable rings R; and (c) connections between cohomology and representation theory for discrete subgroups of Lie groups (**[6]**, **[9]**, **[40]**).

Finally, the reader is referred to the forthcoming proceedings of the 1977 Durham conference on homological and combinatorial techniques in group theory (C. T. C. Wall, ed.) for additional references and a list of open problems.

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