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An infinite-dimensional torsion-free FP_∞ group

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Introduction

A group G is *of type* FP_∞ if there is a projective $\mathbf{Z}G$-resolution $\{P_n\}$ of $\mathbf{Z}$ such that each P_n is finitely generated. G is *of type FP* if $\{P_n\}$ can be chosen so that, in addition, $P_n=0$ for almost all n. Equivalently, G is of type FP if and only if G is of type FP_∞ and G has finite cohomological dimension. A well-known necessary condition for this is that G be torsion free. Problem F 11 of [W; p. 388] asks whether this condition is also sufficient: "Is every torsion-free group of type FP_∞ also of type FP? Or, on the other hand, can one find an FP_∞ group with a non-finitely generated free abelian subgroup?" In this paper we settle this question by showing that a certain group F, previously known to be finitely presented and torsion-free and to have a free abelian subgroup of infinite rank, is of type FP_∞.

The group F is defined by the presentation

$$\langle x_0, x_1, x_2, \dots \mid x_i^{-1} x_n x_i = x_{n+1} \text{ for } i<n \rangle.$$

This group has an interesting history. It was discovered by R.J. Thompson in 1965 in connection with his work in logic – see [MT; p. 475ff]. Thompson also used F in unpublished work as an aid in the construction of some finitely presented infinite simple groups. [We have been able to deduce from the FP_∞ property for F that one of these simple groups is also of type FP_∞; details will appear elsewhere.] The group F later appeared independently in homotopy theory, in connection with the study of homotopy idempotents ([FH], [D_1], [D_2]).

Our proof that F is of type FP_∞ is motivated by this connection with homotopy theory. And as a corollary of the proof we obtain new information about homotopy idempotents, as well as a new proof of the Hastings-Heller theorem [HH] that homotopy idempotents on finite-dimensional complexes split. We also obtain homology and cohomology calculations of F, including the result that $H^*(F, \mathbf{Z}F)=0$. F seems to be the first known FP_∞ group with

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this property. [We will show elsewhere that the simple FP_∞ group mentioned above also has this property.]

The paper is organized as follows. In §1 we summarize known properties of F, emphasizing the point of view that F admits the universal example of an endomorphism which is idempotent up to conjugacy. In §2 we give a heuristic discussion of the topological analogue of F; this is a space Y admitting the universal example of a self map which is idempotent up to homotopy. The rigorous construction of Y is given in §3. We obtain Y as X/F, where X is a space with a free right F-action. It is possible to read §3 independently of §2, but the construction of X will then seem very strange.

We prove in §4 that X is contractible, so that Y is an Eilenberg-MacLane space $K(F,1)$. In §5 we show that Y has the homotopy type of a complex Z of finite type, i.e., with only finitely many cells in each dimension. This proves that F is of type FP_∞. In fact, the cellular chain complex P of the universal cover of Z is a finite type free resolution of $\mathbf{Z}$ over $\mathbf{Z}F$.

We give in §6 a purely algebraic description of this chain complex P. This *description* can be read independently of the rest of the paper. But the proof that P is in fact a resolution relies on the considerations of §§3–5, and we know of no way to avoid this.

In §7 we use the results of the previous sections to calculate $H_*(F,\mathbf{Z})$ and $H^*(F,\mathbf{Z}F)$. This second calculation depends on a result in our paper [BG_2]. Finally, we give in §8 an application of our work to the theory of homotopy idempotents.

We wish to acknowledge our debt to Jerzy Dydak. In unpublished joint work of Dydak and one of us (R.G.) in 1980, there appears a complex Z' of finite type very similar to the space Z mentioned above. It was conjectured at that time that Z' was a $K(F,1)$. The construction of Z' motivated the present work. (In fact, it follows from the work here that Z' *is* a $K(F,1)$.)

This paper and [BG_2] together contain proofs of all theorems announced in [BG_1].

Notational conventions

Given elements a, b of a group, we set $a^b=b^{-1}ab$ and $[a,b]=aba^{-1}b^{-1}$.

All group actions in this paper will be *right* actions. In particular, if G is the group of homeomorphisms of a space X, then the action of G on X is denoted $(x,g)\mapsto xg\,(x\in X, g\in G)$, and composition in G is defined by $x(gh)=(xg)h\,(x\in X, g,h\in G)$.

Given a space Y with basepoint, the composition law in $\pi_1(Y)$ is defined by $[\omega]\cdot[\omega']=[\omega\circ\omega']$, where $\omega\circ\omega'$ traverses ω' followed by ω. This is consistent with the convention of the previous paragraph, in the following sense: Suppose X is the universal cover of Y and G is the group of deck transformations. Then, under the usual hypotheses of covering space theory, there is an isomorphism $\pi_1(Y)\xrightarrow{\approx}G$ which sends $[\omega]$ to the element $g\in G$ such that the lift of ω starting at the basepoint x_0 of X ends at x_0g.

1. The group *F*

Everything in this section can be found in one or more of [MT], [FH], [D_1], [D_2], [DH], [DS]. For the convenience of the reader, we have included proofs of all non-obvious results which will be needed later in the paper.

Recall that F is defined by the presentation

$$\langle x_0, x_1, x_2, \ldots | x_n^{x_i} = x_{n+1} \text{ for } i < n \rangle.$$

It is immediate from this definition that F admits a "shift" map $\phi\colon F \to F$ such that $\phi(x_n) = x_{n+1}$ for $n \geqq 0$ and such that $\phi^2(x) = \phi(x)^{x_0}$ for all $x \in F$; thus ϕ is *idempotent up to conjugacy*. Moreover, the triple (F, ϕ, x_0) is the *universal example* of this situation:

Proposition 1.1. *Given a group G, an endomorphism $\psi\colon G \to G$, and an element $g_0 \in G$ such that $\psi^2(g) = \psi(g)^{g_0}$ for all $g \in G$, there is a unique homomorphism f: $F \to G$ such that $f(x_0) = g_0$ and $f\phi = \psi f$.* $\square$

Remark. Let $g_n = \psi^n(g_0)$ in the situation of 1.1. If $g_1 \neq g_2$, it can be shown that $f\colon F \to G$ is injective, so that the g_n generate a copy of F. More generally, it is known that F admits no nonabelian proper quotients.

Example 1.2. Let g_n ($n \geqq 0$) be the piecewise linear homeomorphism of $\mathbf{R}$ which is the identity on $(-\infty, n]$, has slope 2 on $[n, n+1]$, and has slope 1 on $[n+1, \infty)$. Let S be the homeomorphism of $\mathbf{R}$ given by $uS = u+1$ ($u \in \mathbf{R}$), and note that $g_n^S = g_{n+1}$. The group of homeomorphisms G generated by the g_n therefore admits an endomorphism ψ given by $\psi(g) = g^S$ and satisfying $\psi(g_n) = g_{n+1}$. Note that g_0 agrees with S on $[1, \infty)$; since $\psi(g)$ has support in $[1, \infty)$ for every $g \in G$, it follows that $\psi^2(g) = \psi(g)^S = \psi(g)^{g_0}$. The proposition therefore yields a homomorphism $f\colon F \to G$ such that $f(x_n) = g_n$. [In view of the remark above, f is in fact an isomorphism. But an independent proof of this fact will be given below.]

We will use this homomorphism f to deduce a number of properties of F.

(1.3) **Normal forms.** The relations defining F allow one to write any $x \in F$ in the form $x = x_{i_1} \ldots x_{i_k} x_{j_m}^{-1} \ldots x_{j_1}^{-1}$ with $i_1 \leqq \ldots \leqq i_k$, $j_1 \leqq \ldots \leqq j_m$, k, $m \geqq 0$. Moreover, we can choose this expression for x so that if x_i and x_i^{-1} both occur for some i then x_{i+1} or x_{i+1}^{-1} also occurs. [Otherwise there would be a subproduct of the form $x_i \phi^{i+2}(y) x_i^{-1}$, which could be replaced by $\phi^{i+1}(y)$.] An expression of this form is called a *normal form* for x. We claim that x admits a *unique* normal form.

Proof. It suffices to show that any $g \in G$ has a unique normal form in terms of the g_n. Suppose first that g has a normal form $g_{i_1} \ldots g_{i_k} g_{j_m}^{-1} \ldots g_{j_1}^{-1}$ as above, with all subscripts $\geqq$ some integer i. Then the right-hand derivative of g at $u = i$ is 2^n, where n is the g_i-exponent sum in the normal form. In particular, any other normal form for g with subscripts $\geqq i$ will have the same g_i-exponent sum.

Suppose now that there are two different normal forms for the same element of G, and choose such a pair of normal forms of minimal total length. That total length is necessarily >0. Let i be the smallest subscript occurring in either normal form. The normal forms cannot both start with g_i, since we could then cancel g_i, contradicting the minimality. Similarly, they cannot both end with g_i^{-1}. Since they have the same g_i-exponent sum by the previous paragraph, the only possibility is that one of the normal forms involves both g_i and g_i^{-1} and the other involves neither. The equality between the homeomorphisms represented by the normal forms therefore reads $g_i z g_i^{-1}=w$, where z is a normal form with subscripts $\geqq i$ and w is a normal form with subscripts $\geqq i+1$. Then $z=w^{g_i}=\psi(w)$ as homeomorphisms, hence also as formal expressions by the minimality of our supposed counter-example. But then z involves only subscripts $\geqq i+2$, contradicting the fact that the expression $g_i z g_i^{-1}$ is a normal form. $\square$

As a corollary of the proof, we have:

Corollary 1.4. *The homomorphism f: $F\to G$ of* 1.2 *is an isomorphism.* $\square$

The group G is torsion-free, and ψ: $G\to G$ is injective. Hence:

Corollary 1.5. *F is torsion-free.* $\square$

Corollary 1.6. *ϕ: $F\to F$ is injective.* $\square$

Once ϕ is known to be injective, we see that $F_1\equiv\text{image}(\phi)$ is a copy of F with presentation $\langle x_1, x_2, \ldots | x_n^{x_i}=x_{n+1}$ for $1\leqq i<n\rangle$. A glance at the defining presentation of F now yields:

Proposition 1.7. *F is the HNN extension of F_1 with respect to the monomorphism $(\phi|F_1)$: $F_1\to F_1$, with x_0 as the stable letter.* $\square$

Repeating with respect to $F_2\equiv\text{image}(\phi^2)$, etc., we see that F is an infinitely iterated HNN extension.

Proposition 1.8. *F contains a free abelian subgroup of infinite rank.*

Proof. The elements $x_0x_1^{-1}, x_2x_3^{-1}, x_4x_5^{-1}, \ldots,$ represented as homeomorphisms of $\mathbf{R}$, have disjoint supports. They therefore commute and are linearly independent. $\square$

(1.9) **Finite presentation.** Two finite presentations of F have appeared in the literature. Both have two generators x_0 and x_1. Let x_n for $n\geqq 2$ be the word in x_0 and x_1 defined inductively by $x_n=x_{n-1}^{x_{n-2}}$. Then the two presentations are

$$\langle x_0, x_1 | x_2^{x_0}=x_3,\ x_3^{x_1}=x_4\rangle \quad\text{and}\quad \langle x_0, x_1 | x_2^{x_0}=x_3,\ x_3^{x_0}=x_3^{x_1}\rangle.$$

The relations may also be written as $r_1=r_2=1$ in the first presentation and $r_1=r_3=1$ in the second, where $r_1=[x_0x_1^{-1}, x_2]$, $r_2=[x_1x_2^{-1}, x_3]$, and $r_3=[x_0x_1^{-1}, x_3]$. These three relators have lengths 10, 18, and 14 when written out in terms of x_0 and x_1.

It is not difficult to verify directly that the two presentations above define F. The first presentation will also come out of our work in §5 below, where we will exhibit a $K(F,1)$-complex whose 2-skeleton corresponds to that presentation.

2. The space Y: heuristics

The topological analogue of a conjugacy idempotent on a group is a *homotopy idempotent* on a space, i.e., a based space (Y, y_0) together with a basepoint-preserving map ψ: $Y \to Y$ and a free homotopy $h = (h_t)_{0 \leqq t \leqq 1}$ with $h_0 = \psi^2$ and $h_1 = \psi$. ["Free" means that h_t is *not* required to preserve the basepoint.] Such a triple (Y, ψ, h) gives rise to a conjugacy idempotent $\psi_\#$: $\pi_1(Y) \to \pi_1(Y)$ with $\psi_\#^2([\omega]) = \psi_\#([\omega])^{[\omega_0]}$, where ω_0 is the path $h(y_0)$ (i.e., $t \mapsto h_t(y_0)$).

The topological analogue of the group F is the space Y which admits the universal example of a ψ and h as above:

Proposition 2.1. *There exists a homotopy idempotent (Y, ψ, h) with the following property: Given any other homotopy idempotent (Y', ψ', h'), there is a unique basepoint-preserving map f: $Y \to Y'$ such that $f\psi = \psi' f$ and $fh_t = h'_t f$.*

Sketch of proof. Start with a point y_0. Then attach a 1-cell ω_0 [to be $h(y_0)$] and infinitely many additional cells ω_n [to be $\psi^n(\omega_0)$, $n \geqq 1$]. For each $n \geqq 0$, attach a 2-cell $e_{0,n}$ [to be $h(\omega_n)$, i.e., the 2-cell traced out by $h_t(\omega_n)$, $0 \leqq t \leqq 1$], with the attaching map indicated in the following picture:

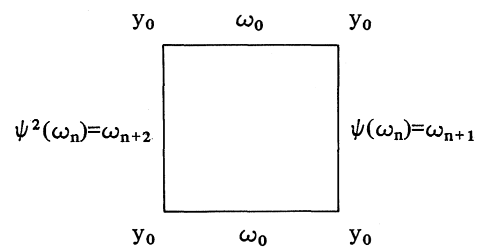

Fig. 1.

[The horizontal coordinate here is the homotopy parameter t.] Then attach infinitely many additional 2-cells $e_{m,n}$ [to be $\psi^m(e_{0,n})$, $m \geqq 1$]. It is not hard to continue in this way; the resulting Y is an infinite-dimensional CW-complex with one n-cell for every n-tuple of non-negative integers. □

For our purposes it will not be necessary to fill in the details of this sketch, since it will be more convenient to construct the universal cover X of Y directly; we will then obtain Y by dividing out by the group of deck transformations. The rest of this section will be devoted to a heuristic discussion of what X should look like, in order to motivate the construction of X to be given in the next section.

Note first that the desired Y has $\pi_1 = F$. In fact, the 1-cell ω_n corresponds to the generator x_n of F; the 2-cell $e_{0,n}$ pictured above corresponds to the relation $x_{n+1}^{x_0} = x_{n+2}$ $(n \geqq 0)$; and the 2-cell $e_{m,n}$ $[= \psi^m(e_{0,n})]$ corresponds to the relation $x_{m+n+1}^{x_m} = x_{m+n+2}$. So X admits a free right F-action, and we may identify the vertex set of X with F.

The 1-cell ω_n of Y lifts to a 1-cell of X going from the vertex 1 to $1 \cdot x_n = x_n$. Denote the points of this lift by x_n^t, $0 \leqq t \leqq 1$. Then the 1-skeleton of X consists of points $x_n^t x$, $x \in F$.

The 2-cell $e_{0,n}$ pictured above has a lift with lower left-hand corner at 1:

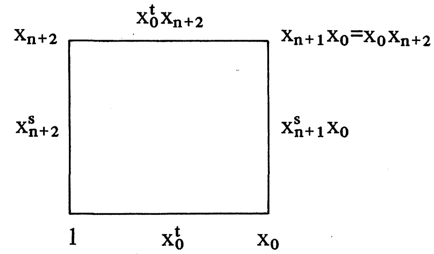

Fig. 2.

Denote the points of this lift by $x_0^t x_{n+2}^s$ ($t, s \in [0, 1]$), with the understanding that on the right-hand edge $t=1$ we have $x_0 x_{n+2}^s = x_{n+1}^s x_0$. Similarly, denote the points of the lift of $e_{m,n}$ by $x_m^t x_{m+n+2}^s$. Then the 2-skeleton of X consists of points $x_m^t x_{m+n+2}^s x$, $m, n \geqq 0$, $t, s \in [0, 1]$, $x \in F$.

We could continue in this way with the 3-cells, etc., but we have already done enough to suggest the following:

(a) The points of X should be represented as products of symbols x_n^t ($n \geqq 0$, $0 \leqq t \leqq 1$ or $t=-1$) satisfying relations analogous to those defining F (such as $x_0^{-1} x_n^t x_0 = x_{n+1}^t$ for $n>0$).

(b) The homotopy idempotent $\psi\colon Y \to Y$ should lift to a map $\phi\colon X \to X$ which preserves products and satisfies $\phi(x_n^t) = x_{n+1}^t$.

(c) The homotopy h on Y should lift to a homotopy on X given by $(x, t) \mapsto x_0^t \phi^2(x)$ ($x \in X$, $0 \leqq t \leqq 1$). [Note that this homotopy goes from ϕ^2 to the map $x \mapsto x_0 \phi^2(x) = \phi(x) x_0$.]

With this motivation, we now proceed to the rigorous construction of X and Y.

3. The space X

Let us identify F with the group of homeomorphisms of $\mathbf{R}$ called G in 1.2 (cf. 1.4). For $0 \leqq t \leqq 1$ and $n \geqq 0$, let x_n^t be the piecewise linear homeomorphism of $\mathbf{R}$ which is the identity on $(-\infty, n]$, has slope 2^t on $[n, n+1]$, and has slope 1 on $[n+1, \infty)$. Let M be the monoid generated by F and the x_n^t; it is a submonoid of the group of homeomorphisms of $\mathbf{R}$.

One easily checks:

Proposition 3.1. $x_i^{-1} x_n^t x_i = x_{n+1}^t$. □

Proposition 3.2. *The shift map $\phi\colon F \to F$ extends to a monoid monomorphism $\phi\colon M \to M$ given by $\phi(x) = x^S$, where S is as in* 1.2; $\phi(x_n^t) = x_{n+1}^t$. □

A *normal form* for $x \in M$ is an expression of x as a product $x_{i_1}^{t_1} \dots x_{i_k}^{t_k} x_{j_m}^{-1} \dots x_{j_1}^{-1}$, where $i_1 \leqq i_2 \leqq \dots \leqq i_k$, $j_1 \leqq \dots \leqq j_m$, $0 < t_r \leqq 1$, satisfying:

(i) among all occurrences of a given x_i with positive exponent, at most one has non-integral exponent and this one occurs first (leftmost);

(ii) if x_i^{+1} and x_i^{-1} both occur, then x_{i+1}^t also occurs for some t with $0 < t \leqq 1$ or $t = -1$.

The proof of the normal form theorem for F (cf. 1.3) goes through without essential change to yield:

Theorem 3.3. *If $x \in M$ admits a normal form, then it admits a unique normal form.* $\square$

The integer $k+m$ above is called the *length* of x and is denoted $l(x)$.

Now let X be the smallest subset of M which contains 1, is closed under ϕ, is closed under right multiplication by F, and satisfies

$$x \in X \Rightarrow x_0^t \phi^2(x) \in X \quad \text{for } 0 \leqq t \leqq 1.$$

[See the end of §2 for the motivation for this.] Equivalently, X can be described as the set of all products $x_{q_1}^{t_1} \dots x_{q_n}^{t_n} x$, with $q_1 \geqq 0$, $q_{i+1} \geqq q_i + 2$, $0 \leqq t_i \leqq 1$, and $x \in F$. It is easy to see that every element of X admits a normal form and hence a unique normal form.

We wish to give X a CW-complex structure. For each n-tuple $\mathbf{q} = (q_1, \dots, q_n)$ as above and each $x \in F$, let $\mathbf{q}x = \{x_{q_1}^{t_1} \dots x_{q_n}^{t_n} x \mid 0 \leqq t_i \leqq 1\}$. Call $\mathbf{q}x$ an *n-cube*. Thus X is the union of cubes. The cubes have faces. We obtain the *A_i-face* [resp. the *B_i-face*] of $\mathbf{q}x$ $(1 \leqq i \leqq n)$ by freezing the exponent of x_{q_i} at 0 [resp. at 1]. Thus (using 3.1)

$$A_i(q_1, \dots, q_n)x = (q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n)x$$
$$B_i(q_1, \dots, q_n)x = (q_1, \dots, q_{i-1}, q_{i+1} - 1, \dots, q_n - 1)x_{q_i} x.$$

Associated to each n-cube $\mathbf{q}x$ is a map $f\colon I^n \to \mathbf{q}x$, $f(t_1, \dots, t_n) = x_{q_1}^{t_1} \dots x_{q_n}^{t_n} x$. By repeated use of Theorem 3.3 the reader can easily check: (i) $f \mid \mathring{I}^n$ is $1-1$. (ii) X is the disjoint union of the images $f(\mathring{I}^n)$, where $\mathbf{q}x$ ranges over all the cubes of X. (iii) The restriction of f to a codimension 1 face of I^n is the map associated to an $(n-1)$-cube of X.

From (i)–(iii) one can deduce:

Proposition 3.4. *X admits a CW-complex structure with the maps $f\colon I^n \to X$ as characteristic maps for the cells.* $\square$

F acts by right multiplication on X as a group of homeomorphisms which freely permute the cubes. It follows that the quotient map $X \to X/F$ is a regular covering map (cf. [Br; §I.4, Exercise 2]).

Remark. It is more or less obvious from what we did above that X is the geometric realization of an abstract semi-cubical complex. But in fact much more is true. One can show that each characteristic map f is $1-1$ and that any two cubes intersect in a (possibly empty) common face. So X is a *cubical complex*, i.e., the cubical analogue of a simplicial complex.

Finally, we complete the circle of ideas begun in §2 by noting that X can be used to give a rigorous proof of 2.1. Let $Y = X/F$, let $\psi\colon Y \to Y$ be the map induced by $\phi\colon X \to X$, and let $h_t\colon Y \to Y$ be induced by $x \mapsto x_0^t \phi^2(x)$. Let the basepoint y_0 of Y be the unique vertex. Then it is easily verified that (Y, ψ, h) has the universal property required in 2.1. Indeed, the construction of $f\colon Y \to Y'$ reduces to the construction of a map $\tilde{f}\colon X \to Y'$ satisfying (i) $\tilde{f}(1) = y'_0$; (ii) $\tilde{f}(xg)$

$=\tilde{f}(x)$ for $x \in X$, $g \in F$; (iii) $\tilde{f}(\phi(x)) = \psi'(\tilde{f}(x))$; and (iv) $\tilde{f}(x_0^t \phi^2(x)) = h_t'(\tilde{f}(x))$. The existence and uniqueness of $\tilde{f}(x)$ is easily proved by induction on the length of the normal form of x, and the continuity of $\tilde{f}$ on a cube $\{x_{q_1}^{t_1} \dots x_{q_n}^{t_n} x\}$ follows trivially from (ii)–(iv) by induction on n.

4. Contractibility of X

Let M be the monoid considered in § 3 and let M_0 be the set of elements of M which admit a normal form. For each $x \in M_0$ we will construct a path $t \mapsto \omega_x(t)$ in M, $0 \leqq t \leqq \infty$, such that $\omega_x(0) = x$ and $\omega_x(t) = 1$ for sufficiently large t. [Note: We have not topologized M, so there is no question of continuity here.]

The idea behind the definition of ω_x is to move x to 1 step by step, where each step consists of (a) decreasing the first exponent in the normal form to 0 or (b) increasing it to 1 and reducing to normal form. We always choose (b) if possible (i.e., if increasing the first exponent to 1 results in an unreduced expression); and in considering this possibility, we allow ourselves to first augment the normal form by adjoining a factor x_i^0 if that is convenient. [Example: If $x = x_1 x_0^{-1}$, then the first leg of ω_x is $t \mapsto x_1^{1-t} x_0^{-1}$, $0 \leqq t \leqq 1$, going from $x_1 x_0^{-1}$ to x_0^{-1}. But if $x = x_2 x_0^{-1} = x_0^0 x_2 x_0^{-1}$, then the first leg of ω_x is $t \mapsto x_0^t x_2 x_0^{-1}$, $0 \leqq t \leqq 1$, going from $x_2 x_0^{-1}$ to $x_0 x_2 x_0^{-1} = x_1$.] The precise definition is as follows.

Let $\omega_1(t) = 1$ for all t. For $x \neq 1$, assume inductively that ω_y has been defined for all $y \in M_0$ with $l(y) < l(x)$. Suppose there is an integer $i \geqq 0$ such that $x = x_i^s \phi^{i+2}(z) x_i^{-1} w$, where $0 \leqq s < 1$, $z \in M_0$, and $w = x_{j_m}^{-1} \dots x_{j_1}^{-1}$ $(0 \leqq j_1 \leqq \dots \leqq j_m \leqq i$, $m \geqq 0)$. Choose the smallest such i and let $y = x_i \phi^{i+2}(z) x_i^{-1} w = \phi^{i+1}(z) w$. Then $l(y) < l(x)$ and we set

$$\omega_x(t) = \begin{cases} s_i^{s+t} \phi^{i+2}(z) x_i^{-1} w & 0 \leqq t \leqq 1-s \\ \omega_y(t-(1-s)) & 1-s \leqq t \leqq \infty. \end{cases}$$

If there is no such i, then the normal form of x necessarily starts with x_j^s for some $j \geqq 0$, $0 < s \leqq 1$. Write $x = x_j^s y$ and set

$$\omega_x(t) = \begin{cases} x_j^{s-t} y & 0 \leqq t \leqq s \\ \omega_y(t-s) & s \leqq t \leqq \infty. \end{cases}$$

We will use the paths ω_x to prove:

Theorem 4.1. *The space X constructed in § 3 is contractible.*

Proof. It suffices to show for any cube C of X that $\omega_z(t) \in X$ for all $(z, t) \in C \times [0, \infty]$ and that the resulting map $C \times [0, \infty] \to X$ is continuous. Let $C = \mathbf{q} x$, where $\mathbf{q} = (q_1, \dots, q_n)$ and $x \in F$. We will argue by induction on $l(C) \equiv n + l(x)$.

We may assume that $l(C) > 0$ and that the result has been proved for cubes C' with $l(C') < l(C)$. We may also assume (by applying ϕ^{-1} if necessary) that C is not in $\operatorname{im}(\phi)$, i.e., that either $q_1 = 0$ or else the normal form of x involves x_0 or x_0^{-1}. Given $(t_1, \dots, t_n) \in I^n$, write $y = y(t_1, \dots, t_n) = x_{q_1}^{t_1} \dots x_{q_n}^{t_n} x$. We must show that $\omega_y(t)$ is in X and varies continuously as a function of the $n+1$ variables

$t, t_1, \ldots, t_n$. Let i be the smallest integer such that either x_i^{+1} occurs in x or x_i^{-1} fails to occur in x. There are two cases, each with two subcases.

Case 1. $n=0$ or $i<q_1$.

Subcase (a). x_i^{+1} occurs in x. Write x as a reduced product $x_i w$ (i.e., the normal form of x is x_i followed by the normal form of w). The normal form of w involves no x_j^{+1} with $j<i$ and every x_j^{-1} with $j<i$. Then $y=x_i y'$, where $y'=x_{q_1+1}^{t_1}\ldots x_{q_n+1}^{t_n} w$, and one checks that

$$\omega_y(t)=\begin{cases} x_i^{1-t} y' & 0\leqq t\leqq 1 \\ \omega_{y'}(t-1) & 1\leqq t\leqq \infty. \end{cases}$$

Note that y' is in the n-cube $C'=(q_1+1, \ldots, q_n+1)w$ and that $x_i^{1-t} y'$ is in the $(n+1)$-cube $k(C)=(i, q_1+1, \ldots, q_n+1)w$. Since $l(C')=l(C)-1$, the induction hypothesis implies that $\omega_y(t)$ is in X and is continuous in $t, t_1, \ldots, t_n$. [For future reference we note that $C=B_1 k(C)$ and that C' is the opposite face $A_1 k(C)$. Our "flow" ω for $0\leqq t\leqq 1$ moves C across $k(C)$ to C'.]

Subcase (b). x_i^{+1} does not occur in x. Then the normal form of x involves x_j^{-1} for all $j<i$ and no other $x_j^{\pm 1}$ with $j\leqq i$. Moreover, our assumptions (including the assumption $C\nsubseteq \operatorname{im}(\phi)$) imply $i\geqq 1$. One checks that

$$\omega_y(t)=\begin{cases} x_{i-1}^{t} y & 0\leqq t\leqq 1 \\ \omega_{y'}(t-1) & 1\leqq t\leqq \infty, \end{cases}$$

where $y'=x_{i-1} y=x_{q_1-1}^{t_1}\ldots x_{q_n-1}^{t_n} x_{i-1} x$. Note that $y'\in C'=(q_1-1, \ldots, q_n-1)x_{i-1}x$, and that $x_{i-1}^{t} y\in k(C)=(i-1, q_1, \ldots, q_n)x$. Since $l(C')=l(C)-1$, the desired result follows from the induction hypothesis. [Note that this time $C=A_1 k(C)$ and $C'=B_1 k(C)$; ω for $0\leqq t\leqq 1$ moves C across $k(C)$ to C', as in (a).]

Case 2. $n>0$ and $i\geqq q_1$. Thus x involves x_j^{-1} and omits x_j^{+1} for all $j<q_1$.

Subcase (a). $x_{q_1} x$ is reduced. Then $B_1 C=(q_2-1, \ldots, q_n-1)x_{q_1} x$ is a cube of the type considered in 1(a) above, and $C=k(B_1 C)$. Hence ω begins by collapsing C onto its face $A_1 C$:

$$\omega_y(t)=\begin{cases} x_{q_1}^{t_1-t} x_{q_2}^{t_2}\ldots x_{q_n}^{t_n} x & 0\leqq t\leqq t_1 \\ \omega_{y'}(t-t_1) & t_1\leqq t\leqq \infty, \end{cases}$$

where $y'=x_{q_2}^{t_2}\ldots x_{q_n}^{t_n} x\in A_1 C=(q_2, \ldots, q_n)x$. Since $l(A_1 C)=l(C)-1$, the result follows from the induction hypothesis.

Subcase (b). $x_{q_1} x$ is not reduced. Then $A_1 C=(q_2, \ldots, q_n)x$ is a cube of the type considered in 1(b), and $C=k(A_1 C)$. Hence ω begins by collapsing C onto $B_1 C$:

$$\omega_y(t)=\begin{cases} x_{q_1}^{t_1+t} x_{q_2}^{t_2}\ldots x_{q_n}^{t_n} x & 0\leqq t\leqq 1-t_1 \\ \omega_{y'}(t-(1-t_1)) & 1-t_1\leqq t\leqq \infty, \end{cases}$$

where

$$y' = x_{q_1} x_{q_2}^{t_2} \dots x_{q_n}^{t_n} x = x_{q_2-1}^{t_2} \dots x_{q_n-1}^{t_n} x_{q_1} x \in B_1 C = (q_2 - 1, \dots, q_n - 1) x_{q_1} x.$$

Since $l(B_1 C) = l(C) - 2$, the proof is complete. □

Corollary 4.2. *The quotient complex $Y = X/F$ is a $K(F,1)$-complex.* □

5. Proof that Y has finite type

The $K(F,1)$-complex $Y = X/F$ that we have just constructed has one n-cell for every n-tuple $\mathbf{q} = (q_1, \dots, q_n)$ of integers with $q_1 \geqq 0$ and $q_{i+1} \geqq q_i + 2$ for $i = 1, \dots, n-1$. We denote that n-cell by $\mathbf{q}$. The face operators on the cubes of X induce operators on the cells of Y:

$$A_i(q_1, \dots, q_n) = (q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n)$$
$$B_i(q_1, \dots, q_n) = (q_1, \dots, q_{i-1}, q_{i+1} - 1, \dots, q_n - 1),$$

$i = 1, \dots, n$.

The interested reader can check that the 2-skeleton of Y is the 2-complex associated to the defining presentation of F: The unique 0-tuple ( ) is the only vertex; the 1-cell (n) corresponds to the generator x_n of F; and the 2-cell $(i, n+1)$ for $i < n$ corresponds to the relation $x_n^{x_i} = x_{n+1}$.

The first finite presentation of F given in 1.9 suggests that the only "essential" cells in dimensions 1 and 2 are the cells (0), (1), (0, 3), and (1, 4). The other 1-cells are "redundant", as are the other 2-cells, except for the 2-cells $(q, q+2)$, $q \geqq 0$; these correspond to the definitions $x_{q+2} = x_{q+1}^{x_q}$ in 1.9. Note that if we start with the unique vertex of Y and the two essential 1-cells and then successively adjoin the 2-cells $(q, q+2)$, $q = 0, 1, \dots$, then each such adjunction is an elementary expansion in the usual sense of simple homotopy theory. We therefore call $(q, q+2)$ a "collapsible" 2-cell and the face $A_1(q, q+2) = (q+2)$ its "free face".

We now extend these ideas to higher dimensions and use them to prove that Y has the homotopy type of a complex with only two cells in each positive dimension.

Fix $n > 0$. The two n-cells $(0, 3, 6, \dots, 3n-3)$ and $(1, 4, 7, \dots, 3n-2)$ are called *essential*. An n-cell $\mathbf{q}$ is called *collapsible* if there is an i $(1 \leqq i < n)$ such that $q_{i+1} - q_i \neq 3$ and, for the largest such i, $q_{i+1} - q_i = 2$; the face $A_i \mathbf{q}$ for this largest i is called the *free face of* $\mathbf{q}$. Those n-cells which are neither essential nor collapsible are called *redundant*.

The following two lemmas are easily checked.

Lemma 5.1. *The free face of a collapsible n-cell is redundant. All of the other faces are either collapsible or else precede the free face in the lexicographic ordering of the $(n-1)$-cells.* □

Lemma 5.2. *The function which sends a collapsible n-cell to its free face is a bijection between the collapsible n-cells and the redundant $(n-1)$-cells.* □

We can now prove the main result of this section.

Theorem 5.3. *Y is homotopy equivalent to a CW-complex with one vertex and with exactly two cells in each positive dimension.*

Proof. Let Y^n be the n-skeleton of Y. Let $Y^0_- = Y^0_+ = Y^0$. For $n>0$, define

$$Y^n_- = Y^{n-1}_+ \cup (\text{essential } n\text{-cells})$$
$$Y^n_+ = Y^n_- \cup (\text{redundant } n\text{-cells}) \cup (\text{collapsible } (n+1)\text{-cells}).$$

Thus $Y^{n-1}_+ \subset Y^n_- \subset Y^n \subset Y^n_+$. We claim that Y^n_+ can be obtained from Y^n_- by a transfinite sequence of elementary expansions. Indeed, we need only adjoin the redundant n-cells in lexicographic order and, along with each redundant n-cell $\mathbf{q}$, the unique collapsible $(n+1)$-cell $\mathbf{q}'$ of which $\mathbf{q}$ is the free face. [Note: Adjoining cells "in lexicographic order" makes sense because the lexicographic ordering is a well-ordering.] By 5.1 all the faces of $\mathbf{q}'$ other than $\mathbf{q}$ have been previously adjoined, so the adjunction of $\mathbf{q}$ and $\mathbf{q}'$ is an elementary expansion, as claimed.

The theorem now follows at once. For future reference, we spell out in detail how the argument above yields a CW-complex Z homotopy equivalent to Y, with only two n-cells for $n>0$. Let $Z^0 = Y^0$. For $n>0$, assume Z^{n-1} has been constructed along with a homotopy equivalence $\pi^{n-1}\colon Y^{n-1}_+ \to Z^{n-1}$. Let $Z^n = Z^{n-1} \cup e^n_0 \cup e^n_1$, where the attaching maps for e^n_0 and e^n_1 are obtained by composing with π^{n-1} the attaching maps for the essential n-cells of Y. Then π^{n-1} extends in an obvious way to a homotopy equivalence $Y^n_- \to Z^n$, and this in turn is extended to a homotopy equivalence $\pi^n\colon Y^n_+ \to Z^n$ via the retraction $Y^n_+ \to Y^n_-$ associated to the elementary expansions above. Passing to the union, we obtain the desired Z and homotopy equivalence $\pi\colon Y \to Z$. □

Corollary 5.4. *The group F is of type FP_∞.* □

6. A finite type resolution

In this section we record an algebraic description of the resolution obtained from the proof in § 5. We begin by giving a direct construction, independent of the previous sections, of a chain complex $P = \{P_n, \partial_n\}_{n \geqq 0}$ of free right $\mathbf{Z}F$-modules, augmented over $\mathbf{Z}$. Let P_0 be free of rank 1, with basis element e^0. For $n>0$ let P_n be free of rank 2 with basis e^n_0, e^n_1.

Proposition 6.1. *There are unique $\mathbf{Z}$-linear maps $\partial = \{\partial_n\colon P_n \to P_{n-1}\}$, $\psi = \{\psi_n\colon P_n \to P_n\}$, $h = \{h_n\colon P_n \to P_{n+1}\}$, and $\varepsilon\colon P_0 \to \mathbf{Z}$ with the following properties:*

(i) *∂ and ε are F-linear; ψ is ϕ-semi-linear* (i.e., $\psi(cx) = \psi(c)\phi(x)$ *for $c \in P$ and $x \in F$, where $\phi\colon F \to F$ is the shift map*); *and h is ϕ^2-semi-linear.*

(ii) *$\partial^2 = 0$, $\varepsilon\partial = 0$, and $\varepsilon(e^0) = 1$.*

(iii) *ψ is a chain map satisfying $\psi(e^0) = e^0$ and $\psi(e^n_0) = e^n_1$ for $n>0$.*

(iv) *Let $\rho_{x_0}\colon P \to P$ be right multiplication by x_0. Then h is a chain homotopy from ψ^2 to $\rho_{x_0}\psi$ satisfying $h(e^0) = e^1_0$, $h(e^n_1) = e^{n+1}_0$ for $n>0$, and $h^2 = 0$.*

Proof. For uniqueness, note first that we must have

$$\begin{aligned} &\varepsilon(e^0)=1, \quad \partial(e^0)=0, \quad \psi(e^0)=e^0, \quad h(e^0)=e_0^1; \\ &\psi(e_0^n)=e_1^n \quad \text{for } n>0; \\ &h(e_0^n)=0, \quad h(e_1^n)=e_0^{n+1} \quad \text{for } n>0. \end{aligned}$$

Next note that $\partial(e_i^n)$ and $\psi(e_1^n)$ are determined inductively. Indeed, using $\partial h + h\partial = \rho_{x_0}\psi - \psi^2$ we find:

$$\begin{aligned} &\partial(e_0^n)=\begin{cases} \partial h(e_1^{n-1})=\psi(e_1^{n-1})x_0-\psi^2(e_1^{n-1})-h\partial(e_1^{n-1}) & n\geqq 2 \\ \partial h(e^0)=\psi(e^0)x_0-\psi^2(e^0)=e^0(x_0-1) & n=1 \end{cases} \\ &\partial(e_1^n)=\partial\psi(e_0^n)=\psi\partial(e_0^n) \\ &\psi(e_1^n)=\psi^2(e_0^n)=\psi(e_0^n)x_0-h\partial(e_0^n)=e_1^n x_0-h\partial(e_0^n). \end{aligned}$$

This completes the proof of uniqueness. For existence, use the formulas above to define the desired maps on basis elements, extend them so that (i) holds, and check that (ii)–(iv) hold. Alternatively, existence will follow from the proof of 6.2 below. □

Theorem 6.2. *The augmented complex P is a free resolution of $\mathbf{Z}$ over $\mathbf{Z}F$.*

Proof. Consider the augmented cellular chain complex $\tilde{P}$ of X. It is a free resolution of $\mathbf{Z}$ over $\mathbf{Z}F$ with one basis element for every cube of X of the form $\mathbf{q}\cdot 1$. Denote this basis element by $\mathbf{q}$. The boundary operator in $\tilde{P}$ is given by the usual formula from cubical homology theory, $\partial=\sum_1^n(-1)^i(A_i-B_i)$. [Note: A_i and B_i are the face operators in X as defined in § 3, not those in Y.] The shift map $\phi\colon X\to X$ induces a chain map $\tilde{\psi}\colon \tilde{P}\to\tilde{P}$ which is ϕ-semi-linear and satisfies $\tilde{\psi}(q_1,\ldots,q_n)=(q_1+1,\ldots,q_n+1)$. The homotopy $I\times X\to X$ given by $(t,x)\mapsto x_0^t\phi^2(x)$ induces a chain homotopy $\tilde{h}\colon\tilde{P}\to\tilde{P}$ which is ϕ^2-semi-linear and satisfies $\tilde{h}(q_1,\ldots,q_n)=(0,q_1+2,\ldots,q_n+2)$. One has $\partial\tilde{h}+\tilde{h}\partial=\rho_{x_0}\tilde{\psi}-\tilde{\psi}^2$.

Now let P' be the cellular chain complex of the universal cover of the complex Z constructed in the proof of Theorem 5.3. It is a resolution which is a quotient complex of $\tilde{P}$. The proof of 5.3 shows that P' has a right $\mathbf{Z}F$-basis given by the images e_0^n and e_1^n of the n-tuples $(0,3,\ldots)$ and $(1,4,\ldots)$ for $n>0$ and the image e^0 of the 0-tuple ( ) for $n=0$. Moreover, the kernel of the quotient map $\tilde{P}\to P'$ is the $\mathbf{Z}F$-sub-chain complex generated by those basis elements $\mathbf{q}$ of $\tilde{P}$ corresponding to the collapsible cells of Y. It follows easily that $\tilde{\psi}$ and $\tilde{h}$ induce maps ψ and h on P' and that properties (i)–(iv) of 6.1 are satisfied. Hence P' can be identified with P, which is therefore a resolution. □

Remark. It is possible to give a purely algebraic proof of the acyclicity of P. One need only go through the proofs of 4.1 and 5.3 and recast them in terms of the chain complex $\tilde{P}$, thereby proving that $\tilde{P}$ is acyclic and that there is a chain homotopy equivalence $\tilde{P}\to P$. On the other hand, we do not know of any *direct* proof of the acyclicity of P, either algebraic or topological, that avoids consideration of the "big" resolution $\tilde{P}$ (or, equivalently, the big $K(F,1)$-complex Y).

7. Homology and cohomology of F

Theorem 7.1. *$H_n(F, \mathbf{Z}) \approx \mathbf{Z} \oplus \mathbf{Z}$ for all $n \geqq 1$, and the shift map $\phi\colon F \to F$ induces a rank 1 idempotent operator on $H_n(F, \mathbb{Z})$.*

Proof. Let $\bar{P}$ be the complex $P \otimes_{\mathbf{Z}F} \mathbf{Z}$ of free $\mathbf{Z}$-modules, where P is the resolution given in §6, and let $\bar{\psi}$ be the endomorphism $\psi \otimes \mathrm{id}_{\mathbf{Z}}$ of $\bar{P}$. Then $H_*(F, \mathbf{Z}) = H_*(\bar{P})$, and $\phi_*\colon H_*(F, \mathbf{Z}) \to H_*(F, \mathbf{Z})$ is the map induced by $\bar{\psi}$ (cf. [Br; §II.6]). The theorem now follows from the formulas given in the proof of 6.1, which show that the boundary operator in $\bar{P}$ is zero and that $\bar{\psi}$ is a rank 1 idempotent in each dimension. □

Theorem 7.2. *$H^n(F, \mathbf{Z}F) = 0$ for all n.*

Proof. Recall from 1.7 that F is an HNN extension with base group $F_1 \approx F$ and associated subgroup F_1. We therefore have a Mayer-Vietoris sequence

$$\ldots \to H^{n-1}(F_1) \to H^n(F) \to H^n(F_1) \xrightarrow{\alpha} H^n(F_1) \to \ldots$$

with an arbitrary F-module of coefficients. It follows from [BG_2] that, because F_1 is of type FP_∞, α is a monomorphism for all n whenever the coefficient module is free over $\mathbf{Z}F$. Now assume inductively that $H^{n-1}(F, L) = 0$ when L is a free $\mathbf{Z}F$-module. Then $H^{n-1}(F_1, L) = 0$ since L is also free over $\mathbf{Z}F_1$, so $H^n(F, L) \approx \ker \alpha = 0$. □

Remarks. 1. F seems to be the first FP_∞ group known to have $H^*(F, \mathbf{Z}F) = 0$.

2. If Z is a $K(F, 1)$ complex of finite type, such as described in the proof of 5.3, the contractible universal cover $\tilde{Z}$ has locally finite skeleta. By a theorem in [M], $\tilde{Z}^n$ is simply connected at ∞, for $n \geqq 2$. The geometrical meaning of Theorem 7.2 is that $\tilde{Z}^n$ is homologically $(n-1)$-connected at ∞: see [GM]. The Hurewicz Theorem for pro-homotopy then implies that $\tilde{Z}^n$ is $(n-1)$-connected at ∞. $Z \times \mathbf{R}^\infty$ is an $\mathbf{R}^\infty$-manifold [H], so $\tilde{Z} \times \mathbf{R}^\infty$, being contractible, is homeomorphic to $\mathbf{R}^\infty$. Thus *F acts freely on $\mathbf{R}^\infty$ equivariantly filtered by finite-dimensional locally finite complexes $\tilde{Z}^n \times I^n$ which are $(n-1)$-connected and $(n-1)$-connected at ∞.* (The relevant infinite-dimensional topology is in [HT] and works referenced there.) It would be interesting to have an explicit description of such an action of F on $\mathbf{R}^\infty$.

8. Homotopy idempotents

In this brief section we record a consequence of the fact that the universal homotopy idempotent (cf. 2.1 and end of §3) lives on a $K(F, 1)$-complex.

Let K be a connected, pointed CW-complex and let $e\colon K \to K$ be a homotopy idempotent. Then $e_\#\colon \pi_1(K) \to \pi_1(K)$ is a conjugacy idempotent, so we get a homomorphism $\rho\colon F \to \pi_1(K)$ by 1.1. The idempotent e is said to be *splittable* if there exist a complex L and maps $K \underset{u}{\overset{d}{\rightleftarrows}} L$ such that $du \simeq \mathrm{id}_L$ and $ud \simeq e$. It is known that this happens if and only if $\ker \rho \neq \{1\}$ (see, for instance, [DS]). In other words, *unsplittable* idempotents are characterized by the fact that they give rise to a copy of F in $\pi_1(K)$.

The canonical example of an unsplittable idempotent is the map $u\colon K(F,1)\to K(F,1)$ induced by $\phi\colon F\to F$. We now know that this canonical example is also the universal example in a sense which we made precise in 2.1. As a consequence, we can show that every unsplittable idempotent e essentially contains a copy of u.

To make this precise, we first pass to the covering space $\tilde{K}$ of K corresponding to image (ρ). Then e lifts to an idempotent $\tilde{e}\colon \tilde{K}\to\tilde{K}$, and $\pi_1(\tilde{K})$ can be identified with F (assuming e is unsplittable). We can now state:

Theorem 8.1. *Let $e\colon K\to K$ be an unsplittable homotopy idempotent and let $\tilde{e}\colon \tilde{K}\to\tilde{K}$ be as above. Then there is a homotopy commutative diagram*

$$\begin{array}{ccccc} K(F,1) & \xrightarrow{\alpha} & \tilde{K} & \xrightarrow{\beta} & K(F,1) \\ \Big\downarrow{\scriptstyle u} & & \Big\downarrow{\scriptstyle \tilde{e}} & & \Big\downarrow{\scriptstyle u} \\ K(F,1) & \xrightarrow{\alpha} & \tilde{K} & \xrightarrow{\beta} & K(F,1) \end{array}$$

such that $\beta\alpha\simeq \mathrm{id}_{K(F,1)}$.

Proof. Taking our space $Y=X/F$ as the model for $K(F,1)$, 2.1 gives us α making the left hand square commute. We get β from the fact that $\pi_1(\tilde{K})=F$, and the right hand square commutes up to homotopy because the induced diagram of fundamental groups commutes. Finally, $\beta\alpha\simeq\mathrm{id}_{K(F,1)}$ for a similar reason. □

Since F has infinite cohomological dimension, $\tilde{K}$ cannot be finite-dimensional in 8.1. Hence we have recovered the following theorem of Hastings and Heller [HH]:

Corollary 8.2. *Any homotopy idempotent on a finite-dimensional complex splits.* □

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