Journal of Pure and Applied Algebra 44 (1987) 77–83
North-Holland

FINITENESS PROPERTIES OF SOLVABLE S-ARITHMETIC GROUPS: AN EXAMPLE

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Communicated by E.M. Friedlander and S. Priddy
Received 4 September 1985

Recall that a group Γ is said to be of *type* FP_n if the $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ admits a projective resolution which is finitely generated in dimensions $\leq n$ (cf. [3] or [6]). For example, Γ is of type FP_1 if and only if it is finitely generated, and Γ is of type FP_2 if it is finitely presented. (The converse of the last assertion is not known.) Stallings [10] gave the first example of a group of type FP_2 but not FP_3, and Bieri [3, §2.6] extended this to a sequence of groups Γ_n of type FP_{n-1} but not FP_n. Further examples of this type can be found in [2], [7], and [12]. The latter shows that such sequences are not particularly exotic; indeed, Stuhler's groups are S-arithmetic groups $\mathbb{SL}_2(\mathcal{O}_S)$, where S is a finite set of primes in a function field and $\mathcal{O}_S$ is the ring of S-integers.

We show here that similar sequences of S-arithmetic groups exist over number fields. Instead of $\mathbb{SL}_2$, however, we must necessarily use a non-reductive algebraic group; for Borel and Serre [5] showed that, in the reductive case, S-arithmetic groups over number fields are always of type FP_∞, i.e., they admit projective resolutions as above which are finitely generated in *all* dimensions.

Our examples are in fact solvable groups. They are of interest for two reasons. First, the question as to which solvable groups are of type FP_n has been quite fruitful and has led to important results about non-solvable groups (cf. Bieri-Neumann-Strebel, "A geometric invariant of discrete groups", preprint). But the question is still wide open, and new examples are needed. Second, our technique of proof relates the question to the Bruhat–Tits building; this may well be the right approach for a systematic investigation.

We now describe our examples. For $n \geq 1$ and p a prime number, let $\Gamma_n \subset \mathbb{GL}_{n+1}(\mathbb{Z}[1/p])$ be the group of upper triangular matrices g with $g_{11} = g_{n+1,\,n+1} = 1$. For instance,

* Partially supported by the National Science Foundation.

$$\Gamma_1 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that Γ_n is an S-arithmetic subgroup of $G_n(\mathbb{Q})$, where $G_n \subset \mathbb{GL}_{n+1}$ is the solvable algebraic matrix group defined by the same equations as Γ_n, and $S = \{p\}$.

Theorem A. *Γ_n is of type FP_{n-1} but not FP_n; for $n \geq 3$ it is finitely presented.*

The negative result (that Γ_n is not of type FP_n) is known and will not be reproved here (cf. [4, Corollary 2 and Proposition]; see also [2]). The finite presentation for $n \geq 3$ is also known, as is the FP_{n-1} property for $n \leq 4$ ([1], [9], [11]). So the new result is that Γ_n is of type FP_{n-1} for $n \geq 5$. But our proof of this includes the previously known positive results with no extra effort.

In view of well-known results about finiteness properties (cf. [7]), the positive part of Theorem A follows from:

Theorem B. *There is a simplicial action of $\Gamma = \Gamma_n$ on a simplicial complex X with the following properties:*

(a) *X is $(n-2)$-connected.*

(b) *The isotropy group of every simplex is finitely presented and of type FP_∞.*

(c) *X is finite mod Γ.*

The remainder of the paper, therefore, will be devoted to Theorem B. In Section 1 we define the complex X; it is a subcomplex of the Bruhat–Tits building associated to the group $\mathbb{GL}_{n+1}$ and the p-adic valuation of $\mathbb{Q}$. In Section 2 we prove (b) and (c) and in Section 3 we prove (a).

Our proof of Theorem B is very specific to the groups Γ_n. An alternate proof, which is somewhat longer but uses more general methods, will be given in [7]. This alternate proof also yields a new proof that Γ_n is not of type FP_n.

1. The complex X

1.1. Notation

K = a field with discrete valuation v.

A = the valuation ring.

π = a generator of the maximal ideal of A.

$V = K^{n+1}, \quad n \geq 0.$

A *lattice* L in V is a finitely generated A-submodule of V spanning V as a vector space over K. Equivalently, L is a free A-submodule of V with an A-basis which is also a vector space basis of V. The set $\mathscr{L}$ of lattices in V is the set of vertices of

an n-dimensional simplicial complex W defined as follows: A simplex in W is a finite subset $\{L_0, \ldots, L_q\}$ of $\mathscr{L}$ such that

$$L_0 < L_1 < \cdots < L_q < \pi^{-1} L_0.$$

For any subset $\mathscr{L}'$ of $\mathscr{L}$ we will denote by $W \mid \mathscr{L}'$ the full subcomplex of W with vertex set $\mathscr{L}'$.

1.2. Assume $n \geq 1$ and let $e_0, \ldots, e_n$ be the standard basis of V. Let $i: K \to V$ be the inclusion of the first factor, $i(\alpha) = \alpha e_0$, and let $p: V \to K$ be the projection onto the last factor, $p(\sum \alpha_i v_i) = \alpha_n$. Let $\mathscr{L}_1 = \{L \in \mathscr{L} : i^{-1}(L) = A\}$ and let $\mathscr{L}_2 = \{L \in \mathscr{L} : p(L) = A\}$. Let $W_j = W \mid \mathscr{L}_j$, $j = 1, 2$. W_1 and W_2 are both isomorphic to the Bruhat–Tits building corresponding to our situation, cf. [8, pp. 77 and 75]. In particular, they are n-dimensional and contractible. Let $X = W_1 \cap W_2$. It is not hard to show that X is $(n-1)$-dimensional, but we will not need this. For $K = \mathbb{Q}$ and v the p-adic valuation, we will show in Sections 2 and 3 that the simplicial complex X has the properties claimed in Theorem B.

We close this section by recalling the proof in [8] that the Bruhat–Tits building (in the form of W_1 or W_2 above) is contractible. We will need a generalization of this (1.3 and 1.4 below), for which the same proof works. Consider first W_2. It is an *ordered* simplicial complex, with the vertices ordered by inclusion. The real line $\mathbb{R}$ is the geometric realization of an ordered simplicial complex, R say, with $\mathbb{Z}$ as vertex set and the usual ordering. We can therefore form the product $W_2 \times R$, which is an ordered simplicial complex whose geometric realization $|W_2 \times R|$ is $|W_2| \times \mathbb{R}$. Letting L_0 be an arbitrary basepoint in W_2, one now defines a simplicial map $F: W_2 \times R \to W_2$ by

$$F(L, n) = \begin{cases} L + \pi^{-n} L_0, & n \leq 0, \\ L_0 + \pi^n L, & n \geq 0, \end{cases}$$

$L \in \mathscr{L}_2$, $n \in \mathbb{Z}$. It is a routine matter to verify that the function F so defined on vertices does indeed extend to a simplicial map. For any vertex L we have $F(L, n) = L$ for $n \ll 0$ and $F(L, n) = L_0$ for $n \gg 0$. It follows easily that W_2 is contractible. See [8] for more details.

The same proof shows:

1.3. Proposition. *Let $\mathscr{L}'$ be a non-empty subset of $\mathscr{L}$ such that for every two lattices $L_1, L_2 \in \mathscr{L}'$ we have*

(1) $L_1 + L_2 \in \mathscr{L}'$, *and*

(2) *if* $L_1 \subset L \subset L_2$, *then* $L \in \mathscr{L}'$.

Then $W \mid \mathscr{L}'$ is contractible.

A similar proof, using intersections instead of sums, shows that W_1 is contractible, and, more generally:

1.4. Proposition. *Let $\mathscr{L}'$ be a non-empty subset of $\mathscr{L}$ such that for every two lattices $L_1, L_2 \in \mathscr{L}'$ we have*

(1) $L_1 \cap L_2 \in \mathscr{L}'$, *and*

(2) *if* $L_1 \subset L \subset L_2$, *then* $L \in \mathscr{L}'$.

Then $W \mid \mathscr{L}'$ *is contractible.*

Note that our complex $X = W_1 \cap W_2$ does not satisfy the hypotheses of either 1.3 or 1.4; but we will need to apply 1.4 to certain subcomplexes of X in Section 3.

2. The action of Γ: Orbits and isotropy groups

Let B be the upper triangular subgroup of $\mathbb{GL}_{n+1}$,

$$B = \{g \in \mathbb{GL}_{n+1} : g_{ij} = 0 \text{ for } i > j\}.$$

2.1. Proposition. (a) *The subgroup $B(K)$ of $\mathbb{GL}_{n+1}(K)$ acts transitively on the set $\mathscr{L}$ of lattices in $V = K^{n+1}$.*

(b) *For $K = \mathbb{Q}$ and v the p-adic valuation, the subgroup $B(\mathbb{Z}[1/p])$ of $\mathbb{GL}_{n+1}(\mathbb{Q})$ acts transitively on $\mathscr{L}$.*

Proof. (a) A basis $v_0, \ldots, v_n$ of V will be called *triangular* (with respect to the standard basis $e_0, \ldots, e_n$) if there is a matrix $\alpha \in B(K)$ such that $v_j = \sum_i \alpha_{ij} e_i$ for every j. A restatement of (a) is that every lattice has a triangular basis. We will prove this by induction on n. If $n = 0$ this follows immediately from the fact that every lattice in K has the form $\pi^m A$. For the inductive step, let V' be the vector space spanned by $e_0, \ldots, e_{n-1}$ and let $L' = L \cap V'$. By induction, L' has a triangular basis with respect to $e_0, \ldots, e_{n-1}$. Let p: $V \to K$ be the projection onto the last factor and choose $v_n \in L$ such that $p(v_n)$ generates the rank-1 lattice $p(L)$ in K. Then $v_0, \ldots, v_n$ is a triangular basis for L.

(b) Suppose $K = \mathbb{Q}$ and v is the p-adic valuation. We have to show that the triangular basis above can be chosen so that $\alpha_{ij} \in \mathbb{Z}[1/p]$. If $n = 0$ this follows from the proof of (a). For the inductive step, the proof of (a) shows that we may take v_n of the form $p^a \cdot e_n + v'$, $a \in \mathbb{Z}$, $v' \in V'$. We may change v' by adding a vector in L'. So it suffices to show that $V' = L' + \sum_{i \le n-1} \mathbb{Z}[1/p] e_i$. By the inductive hypothesis we may apply a matrix in $\mathbb{GL}_n(\mathbb{Z}[1/p])$ to transform L' to the standard lattice $\sum_{i \le n-1} A e_i$. So we may assume L' is the standard lattice, and we are reduced to the well-known (and easy) fact that $\mathbb{Q} = A + \mathbb{Z}[1/p]$.

Proof of Theorem B(c). 2.1(b) implies immediately that Γ acts transitively on the set of vertices of X. Since X is locally finite, (c) follows.

Proof of Theorem B(b). For the standard lattice $L_0 = \sum_{i \le n} A e_i$ the isotropy group

Γ_{L_0} is of finite index in the triangular group $B(\mathbb{Z})$. The latter has a subgroup of finite index which is finitely generated and nilpotent, hence it is finitely presented and of type FP_∞. Since the isotropy group Γ_L of every vertex L is conjugate to Γ_{L_0} (cf. proof of Theorem B(c) above), the same is true of Γ_L. Finally, the isotropy group Γ_σ of a simplex σ is of finite index in the isotropy group of any of its vertices by local finiteness, so Γ_σ is also finitely presented and of type FP_∞.

3. Proof of Theorem B(a): X is $(n-2)$-connected

We will prove a slightly more general result. Let K be as in Section 1, let $p\colon V \to Q$ be a surjective linear map of K-vector spaces, and let U be a subspace of $\ker p$. Let L' and L'' be lattices in U and Q, respectively, and let $\mathscr{L}(L', L'')$ be the set of lattices L in V such that $L \cap U = L'$ and $p(L) = L''$.

3.1. Proposition. *Suppose* $\dim Q = 1$ *and let* $r = \dim V - \dim U - 1$. *Then* $W \mid \mathscr{L}(L', L'')$ *is* $(r-1)$*-connected.*

Note that in the situation of Theorem B we have $X = W \mid \mathscr{L}(L', L'')$, with $\dim U = \dim Q = 1$ and $r - 1 = n - 2$; so 3.1 does indeed yield Theorem B(a). The rest of this section will be devoted to the proof of 3.1. The dimension restriction on Q is not needed for the first lemma:

3.2. Lemma. *Let M be a finitely generated A-submodule of V such that* (a) $M \cap U \subset L'$ *and* (b) $p(M) \subset L''$. *Then there is a lattice in $\mathscr{L}(L', L'')$ containing M.*

Proof. Replacing M by $M + L'$ if necessary, we may assume (a′) $M \cap U = L'$. We may also assume M is a lattice in V. For we can choose in $p^{-1}(L'')$ a basis T for a complement of $\mathrm{span}_K(M)$ in V, and then the lattice $M + \mathrm{span}_A(T)$ still satisfies (a′) and (b).

Assume now that M is a lattice satisfying (a′) and (b). Let $w = \dim Q$ and let $T = \{t_1, \ldots, t_w\}$ be any subset of M mapping to a basis of the lattice $p(M)$. Then $M = (M \cap \ker p) \oplus \mathrm{span}_A(T)$. Let $S = \{s_1, \ldots, s_w\}$ be a basis of L'', and let $\alpha \in \mathbb{GL}_w(K)$ be the matrix such that $s_j = \sum_i \alpha_{ij} p(t_i)$. Then

$$(M \cap \ker p) \oplus \mathrm{span}_A \left\{ \sum_i \alpha_{ij} t_i \right\}_{j=1,\ldots,w}$$

is a lattice containing M [because α^{-1} has entries in A] and lying in $\mathscr{L}(L', L'')$. □

Now suppose $\dim Q = 1$ and let $L'' = A w_0$. Let H be the affine hyperplane $p^{-1}(w_0)$ in V.

3.3. Lemma. $W \mid \mathscr{L}(L', L'')$ *is homotopy equivalent to the simplicial complex* Σ

whose simplices are the finite subsets σ of H such that $\operatorname{span}_A(\sigma) \cap U \subset L'$.

Proof. For any $v \in H$ let $\mathscr{L}_v = \{L \in \mathscr{L}(L', L''): v \in L\}$ and let $Z_v = W \mid \mathscr{L}_v$. Then $W \mid \mathscr{L}(L', L'') = \bigcup_{v \in H} Z_v$. It is easy to see that $\mathscr{L}_v$ satisfies the hypotheses of 1.4, as does any non-empty intersection of $\mathscr{L}_v$'s. Hence Z_v, as well as any non-empty intersection of Z_v's, is contractible. Consequently, $W \mid \mathscr{L}(L', L'')$ has the homotopy type of the nerve of the cover $\{Z_v\}_{v \in H}$ (see, for instance, [8], 1.9). But 3.2 implies that this nerve is precisely Σ. $\square$

It will be convenient to enlarge Σ to the complex $\tilde{\Sigma}$ whose simplices are the finite subsets τ of H such that the r-skeleton of τ is contained in Σ. [Here r is as in 3.1 and the r-skeleton of τ consists of all subsets σ of τ such that $\#\sigma \leq r+1$.] Note that Σ and $\tilde{\Sigma}$ have the same r-skeleton. To complete the proof of 3.1, then, it suffices to prove:

3.4. Proposition. *$\tilde{\Sigma}$ is contractible.*

It is easy to see intuitively why 3.4 should be true: Since $r+1$ is the codimension of U in V, 'most' subsets σ of H with $\#\sigma \leq r+1$ will satisfy $\operatorname{span}_K(\sigma) \cap U = 0$; in particular, they will be simplices of Σ. Hence any subset τ of H 'in general position' will be a simplex of $\tilde{\Sigma}$. This suggests that $\tilde{\Sigma}$ is, up to small perturbation, the same as the obviously contractible complex consisting of *all* finite subsets of H. We now make these ideas precise.

We will say that a subset σ of H is *independent* if it is linearly independent as a subset of V, or, equivalently, if it is affinely independent as a subset of H. We will say that σ is *independent mod U* if it maps to an independent set in V/U, i.e., if $\sum_{v \in \sigma} \lambda_v v \in U$ implies $\lambda_v = 0$ for all v. Finally, we say that a subset S of H is in *general position mod U* if the following equivalent conditions hold:

(1) Every subset σ of S with $\#\sigma \leq r+1$ is independent mod U.

(2) Every subset σ of S with $\#\sigma \leq r+1$ is independent and satisfies $\operatorname{span}_K(\sigma) \cap U = 0$.

For any subset S of H, let $\Sigma \mid S$ (resp. $\tilde{\Sigma} \mid S$) denote the full subcomplex of Σ (resp. $\tilde{\Sigma}$) with vertex set S.

3.5. Lemma. *Let S_1 be a finite subset of H. There is a finite subset S_2 of H with the following properties:*

(a) *S_2 is in general position mod U.*

(b) *There is a simplicial map $\varphi: \tilde{\Sigma} \mid S_1 \to \tilde{\Sigma} \mid S_2$ such that $\sigma \cup \varphi\sigma$ is a simplex of $\tilde{\Sigma}$ for every simplex σ of $\tilde{\Sigma} \mid S_1$.*

3.5 implies 3.4. First note that every finite subset S of H in general position is a simplex in $\tilde{\Sigma}$ by condition (2) above, hence $\tilde{\Sigma} \mid S$ is contractible. Now 3.5(b) implies that the inclusion of $\tilde{\Sigma} \mid S_1$ into $\tilde{\Sigma}$ is homotopic to a map which factors through the

contractible complex $\tilde{\Sigma}|S_2$. Thus every finite subcomplex of $\tilde{\Sigma}$ is null-homotopic in $\tilde{\Sigma}$, so $\tilde{\Sigma}$ is contractible.

Proof of 3.5. For each simplex σ of Σ we can find a lattice L_σ in V such that $\mathrm{span}_A(\sigma)\subset L_\sigma$ and $U\cap L_\sigma\subset L'$ (cf. 3.2). Taking the intersection of the L_σ for all σ in $\Sigma|S_1$, we obtain a lattice L in V such that $U\cap(L+\mathrm{span}_A(\sigma))\subset L'$ for all simplices σ of $\Sigma|S_1$. Write $S_1=\{v_1,\dots,v_t\}$. We will define $S_2=\{w_1,\dots,w_t\}\subset H$ in general position mod U, with $w_i\in v_i+L$, and we will set $\varphi(v_i)=w_i$. The conditions of 3.5 will then be satisfied. Assume inductively that w_j has been defined for $j<i$. In view of condition (1) in the definition of 'general position mod U', we must find $w_i\in(v_i+L)\cap H$ such that w_i is not contained in any of the subspaces $U+\mathrm{span}_K(\sigma)$ for $\sigma\subset\{w_1,\dots,w_{i-1}\}$ with $\#\sigma\le r$. This is possible because $(U+\mathrm{span}_K(\sigma))\cap H$ is a proper affine subspace of H, and a finite set of proper affine subspaces of H cannot cover $(v_i+L)\cap H$.

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