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FIVE LECTURES ON BUILDINGS

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These lectures are intended to provide an introduction to J. Tits's theory of buildings. The point of view is generally the same as that of my book [10], which will be the main reference in what follows. For a different point of view and much more information about buildings, see [18]. See also the conference proceedings [17] and [19] for some examples of recent research.

I have tried to minimize the prerequisites, especially for the first lecture. But from the second lecture on, readers will need to be familiar with Coxeter groups and Coxeter complexes. Appendix A below provides a brief review of some of this material. Readers can refer to it as necessary.

LECTURE 1. DEFINITION AND EXAMPLES

Buildings are simplicial complexes satisfying certain axioms that will be given below. In order to motivate these axioms, we begin with two examples.

1. Examples

Example 1. Let k be a field and let V be the n-dimensional vector space k^n. Let Δ be the simplicial complex associated to the poset of proper non-zero subspaces of V, ordered by inclusion. Thus the vertices of Δ are the proper non-zero subspaces of V, and the simplices are the chains

$$V_1 < \cdots < V_q$$

of such subspaces. The maximal simplices, called *chambers*, are those with $\dim V_i = i$ for all i. They have rank $n-1$ (i.e., they have $n-1$ vertices) and hence dimension $n-2$. Note that, for a suitable choice of basis $e_1, \ldots, e_n$, such a simplex corresponds to the chain

$$[e_1] < [e_1, e_2] < \cdots < [e_1, \ldots, e_{n-1}],$$

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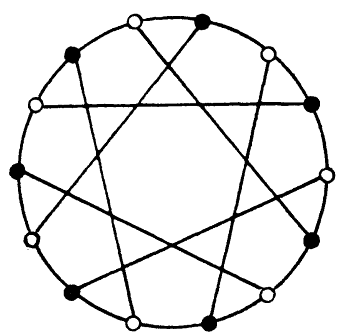
where the square brackets denote the subspace spanned by a set of vectors. We call Δ the *building* associated to V and we write $\Delta = \Delta(V)$.

If $n = 3$, for instance, then $\dim \Delta = 1$, and Δ is simply the incidence graph of lines and planes in k^3 passing through the origin. Thus there is a vertex for each line L, a vertex for each plane P, and an edge joining these vertices whenever $L \subset P$. [Equivalently, Δ is the incidence graph of points and lines in the projective plane over k.]

The picture below shows this graph in case k is the field $\mathbf{F}_2$ with 2 elements. It has 14 vertices and 21 edges. The vertices are shown in two different "colors" to indicate the two possible types of proper non-zero subspaces of k^3 (lines and planes). Note that each vertex has degree 3, i.e., has exactly 3 edges coming out of it. This corresponds to the fact that, over $\mathbf{F}_2$, each plane contains exactly 3 lines through the origin and each line is contained in exactly 3 planes.



Continuing with the case $n = 3$, note that Δ contains lots of hexagons. Indeed, each basis e_1, e_2, e_3 for V yields a configuration

$$\begin{tikzcd}
& & {[e_1]} & & \\
& {[e_1, e_2]} \arrow[ur, dash] & & {[e_1, e_3]} \arrow[ul, dash] & \\
{[e_2]} \arrow[ur, dash] \arrow[rr, dash] & & {[e_2, e_3]} \arrow[rr, dash] & & {[e_3]} \arrow[ul, dash]
\end{tikzcd}$$

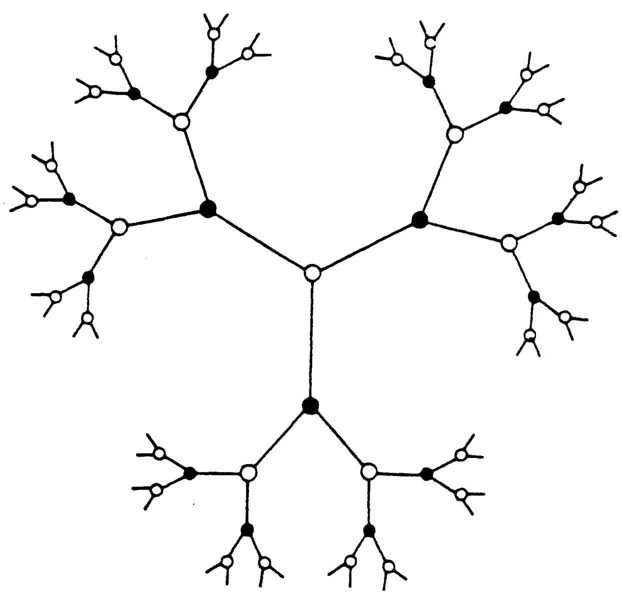
in Δ. These hexagons are called *apartments*. There is one for each unordered triple of independent 1-dimensional subspaces of k^3 [or, equivalently, for each triangle in the projective plane]. In the example with $k = \mathbf{F}_2$, there are 28 apartments. The reader should locate a few of them in the picture.

The phrase "lots of hexagons" above can be made more precise. For example, one can easily find an apartment containing any given chamber. And with only slightly more effort, one can even find an apartment containing any two given chambers. (Try a few examples of this in the picture above.)

Returning now to the case of arbitrary n, it is still true that every basis $e_1, \ldots, e_n$ determines a subcomplex $\Sigma \subset \Delta$, called an *apartment*. It consists of the simplices which can be constructed by using subspaces spanned by subsets of $\{e_1, \ldots, e_n\}$. One can check that Σ is isomorphic to the barycentric subdivision of the boundary of an $(n-1)$-simplex. In particular, Σ is topologically an $(n-2)$-sphere.

And it is still true that any two chambers are contained in an apartment, but this is a more substantial exercise than in the case $n = 3$. See [10], Exercise 2 of §IV.2, where this assertion is deduced from the proof of the Jordan–Hölder theorem. See also Abels ([1], [2]) for generalizations.

Example 2. For our second example of a building, let Δ be a tree in which every vertex has degree ≥ 3. For example, Δ could be the tree pictured below, in which every vertex has degree exactly 3. As in Example 1, the vertices have been drawn in two colors, so that each chamber (edge) has one vertex of each color. Vertices of the same color will be said to have the same *type*. This equivalence relation is intrinsic to Δ; in fact, two vertices have the same type if and only if the distance between them is even.



By an *apartment* in Δ we will mean any subcomplex Σ which is isomorphic to a triangulated line (infinite in both directions). There are uncountably many apartments. Once again, it is easy to see that any two chambers are contained in an apartment.

2. The definition

We need to recall some terminology. Let Δ be a finite-dimensional simplicial complex in which all maximal simplices have the same dimension. Call the maximal simplices *chambers*. Two chambers C, D are *adjacent* if they have a common codimension 1 face. A *gallery* from C to D is a sequence of chambers

$$C = C_0, C_1, \ldots, C_l = D$$

such that C_{i-1} and C_i are adjacent for $i = 1, \ldots, l$. The gallery is said to have *length* l. We call Δ a *chamber complex* if any two chambers can be connected by a gallery. For example, every triangulated manifold is a chamber complex.

A chamber complex is *thin* if every simplex of codimension 1 is a face of exactly two chambers. Thus a thin chamber complex is precisely what combinatorial topologists call a *pseudomanifold without boundary*. In particular, every triangulated manifold without boundary is a thin chamber complex. Finally, a chamber complex is *thick* if every simplex of codimension 1 is a face of at least 3 chambers.

Note that a 1-dimensional chamber complex is nothing but a connected simplicial graph; it is thick if and only if every vertex has degree ≥ 3, and it is thin if and only if every vertex has degree 2 [in which case it is a line or a polygon].

We are ready now for the main definition. The theory is somewhat simpler if we require our buildings to be thick, so we will do that for the moment. At the end of this lecture we will remove that restriction.

Definition (Tits, 1965). Let Δ be a finite-dimensional simplicial complex. We call Δ a (thick) *building* if it can be expressed as the union of a family of subcomplexes Σ, called *apartments*, satisfying the following axioms:

(B0) *Each apartment is a thin chamber complex of the same dimension as* Δ.

(B1) *Any two simplices of* Δ *are contained in an apartment.* [Hence Δ is a chamber complex.]

(B2) *Given two apartments* Σ, Σ' *with a common chamber, there is an isomorphism* $\Sigma \xrightarrow{\approx} \Sigma'$ *fixing* $\Sigma \cap \Sigma'$ *pointwise.*

(B3) Δ *is thick.*

It should be clear from our discussion of Examples 1 and 2 that those examples do in fact satisfy **(B0)**, **(B1)**, and **(B3)**. To see that **(B2)** holds in Example 2, note that the intersection of two lines is a convex subset of each of them, hence an interval (possibly unbounded); it is now a simple matter to construct the desired isomorphism. It is also easy to verify **(B2)** in Example 1 with $n = 3$; one need only think about the possibilities for the intersection of two of our hexagonal apartments. The general case of Example 1 takes a little more work; see [10], Exercise 2 of §IV.2.

One can best get a feeling for these axioms by looking at some consequences of them.

3. Consequences of the axioms

We will omit most of the proofs in this section; they can all be found in Chapter IV of [10]. We begin with an easy observation:

Proposition 1. *All apartments are isomorphic.*

Proof. Given two apartments Σ, Σ', choose by **(B1)** an apartment Σ'' containing a chamber of Σ and a chamber of Σ'. Then **(B2)** yields isomorphisms $\Sigma \xrightarrow{\approx} \Sigma'' \xrightarrow{\approx} \Sigma'$. □

Much less obviously, one can show:

Proposition 2. *The isomorphism type of the apartments is determined by* Δ.

The content of this is the following: Let $\mathcal{A}$ be a system of apartments, i.e., a family of subcomplexes Σ satisfying the axioms. Then for any other system of apartments $\mathcal{A}'$, the complexes $\Sigma \in \mathcal{A}'$ are isomorphic to those in $\mathcal{A}$. If, for instance, Δ has an apartment system consisting of lines (resp. hexagons) then every apartment system consists of lines (resp. hexagons). We will explain why this is true in Lecture 2.

Proposition 3. *If the apartments are finite complexes, then* Δ *admits a unique system of apartments.*

For example, if there is an apartment system $\mathcal{A}$ consisting of hexagons, then $\mathcal{A}$ is the only possible system of apartments in Δ. [One can even show in this case that $\mathcal{A}$ necessarily consists of all of the hexagons in Δ.]

Proposition 3′. *In the general case, there is a unique maximal system of apartments.*

Another way to say this is that any two apartment systems are compatible with one another, in the sense that their union is again an apartment system. [What has to be shown here is that the union satisfies **(B2)**.] In the tree case, for instance, the maximal apartment system is the one we described in Example 2 above, consisting of all lines. But there are also smaller apartment systems, containing only some of the lines; one need only take enough lines to satisfy **(B1)**.

Proposition 4. *The apartments are Coxeter complexes.*

The reader can refer to Appendix A for a review of what this means. For our present purposes, however, it suffices to recall that Coxeter complexes are certain very special thin chamber complexes, associated to "generalized reflection groups". For example, every finite Coxeter complex is topologically a sphere, with a triangulation induced by a finite reflection group. A special case of the proposition, then, is that no closed manifold other than a sphere can ever occur as an apartment in a building. This may seem surprising at first, and it illustrates the force of the axioms.

The idea of the proof of Proposition 4 is the following: Given an apartment Σ, one combines various isomorphisms given by **(B2)** in order to construct automorphisms of Σ which behave like reflections. These "reflections" generate a group W whose action on Σ resembles the action of a finite reflection group on Euclidean space (or on the unit sphere in that space); one deduces that W is a Coxeter group and that Σ is the associated Coxeter complex. The proof uses the thickness axiom **(B3)** in order to construct "enough" reflections. [It is clear *a priori* that thickness has to be used; for if we were to drop **(B3)**, then any thin chamber complex Σ could occur as an apartment: We could simply take $\Delta = \Sigma$, which would then be a building with a single apartment.]

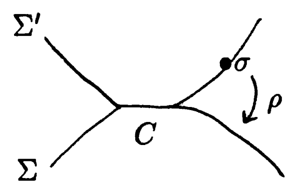
Looking back at the examples, one can check directly that the apartments are Coxeter complexes. In Example 1, the associated Coxeter group is the symmetric group on n letters (or the dihedral group of order 6 if $n = 3$). And in Example 2 it is the infinite dihedral group.

The next result is not about the apartments themselves, but rather about how they sit as subcomplexes of Δ:

Proposition 5. *Every apartment Σ is a retract of Δ and is convex in Δ.*

The word "convex" here is used in a combinatorial sense: For any two chambers of Σ, any gallery of minimal length joining them in Δ is entirely contained in Σ. [Intuitively, a minimal gallery is something like a geodesic, whence the term "convex".]

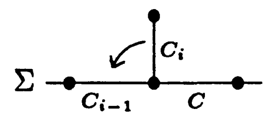
Sketch of Proof. To construct a retraction $\rho : \Delta \to \Sigma$, choose a chamber C of Σ, which will be fixed throughout the construction. Given a simplex σ of Δ, we can find an apartment Σ' containing C and σ. By **(B2)** there is then an isomorphism $\phi : \Sigma' \to \Sigma$ fixing C pointwise, and we set $\rho(\sigma) = \phi(\sigma)$:



It is not hard to check that ρ is a well-defined simplicial map. And it is clearly a retraction, since we can take $\Sigma' = \Sigma$ and $\phi = \mathrm{id}_\Sigma$ if $\sigma \in \Sigma$.

Note, for future reference, that ρ depends on the choice of C but is canonical otherwise. [This assertion requires a little thought, but it is not difficult to check.] We denote ρ by $\rho_{\Sigma,C}$. The retractions $\rho_{\Sigma,C}$ are extremely useful technical tools. They are used in the proofs of many of the results that we have stated above without proof. We will also use them now, to prove the convexity of apartments.

Suppose $C_0, \ldots, C_l$ is a minimal gallery from C_0 to C_l in Δ, with extremities $C_0, C_l \in \Sigma$. Suppose the gallery does not stay in Σ, say $C_{i-1} \in \Sigma$ but $C_i \notin \Sigma$. Let C be the chamber of Σ which is adjacent to C_{i-1} along the same face as C_i, and let ρ be the retraction $\rho_{\Sigma,C}$ constructed above. From the definition of ρ, one can check that $\rho(C_i) = C_{i-1}$:



But then the image of our gallery under ρ is a gallery which has the same extremities but "stutters" [it repeats a chamber]. Removing the repetition, we obtain a shorter gallery from C_0 to C_l, contradicting minimality of the original gallery. $\square$

Here is another application of retractions:

Proposition 6. *Δ is labellable.*

This means that it is possible to partition the vertices into n "types", where $n = \operatorname{rank} \Delta$, in such a way that each chamber has exactly one vertex of each type. (See §3 of Appendix A for more details.) In Example 1, for instance, a vertex is a subspace V' of a vector space V, and its type is determined by its dimension. The proof of labellability is immediate: One knows that Coxeter complexes are labellable (cf. Appendix A, §3), so we need only label one apartment Σ and then extend the labelling to Δ by using a retraction $\rho : \Delta \to \Sigma$.

Finally, we state the *Solomon–Tits theorem*, which shows that the axioms severely limit how complicated the algebraic topology of a building can be:

Proposition 7. *If the apartments are finite (hence $(n-1)$-spheres, where $n = \operatorname{rank} \Delta$), then Δ has the homotopy type of a bouquet of $(n-1)$-spheres. If the apartments are infinite, then Δ is contractible.*

4. The role of thickness

Although thick buildings as defined above are the most important ones for applications, it is sometimes convenient to have a more general notion of "building". It turns out that there is a perfectly satisfactory theory without the thickness axiom, provided we add the assumption that the apartments are Coxeter complexes. In other words, we drop **(B3)** but then take the result of Proposition 4 as a new axiom. All of the results above remain valid if we do this. [The point is that thickness is used only to prove Proposition 4, but the latter is used in many of the other proofs.]

Here, then, is our revised definition:

Definition. Let Δ be a finite-dimensional simplicial complex. We call Δ a *building* if it can be expressed as the union of a family of subcomplexes Σ, called *apartments*, satisfying the following axioms:

(B0) *Each apartment is a Coxeter complex of the same dimension as* Δ.

(B1) *Any two simplices of* Δ *are contained in an apartment.*

(B2) *Given two apartments* Σ, Σ' *with a common chamber, there is an isomorphism* $\Sigma \xrightarrow{\approx} \Sigma'$ *fixing* $\Sigma \cap \Sigma'$ *pointwise.*

Note, for instance, that a building in our new sense can even be thin; indeed, a thin building is precisely the same thing as a Coxeter complex. Note also that we can now generalize Example 2: Any tree in which every vertex has degree ≥ 2 is a building.

Lecture 2. Galleries, etc.

Let Δ be a building and let $\mathcal{C}$ be the set of chambers of Δ. One can view $\mathcal{C}$ as a metric space with the gallery metric

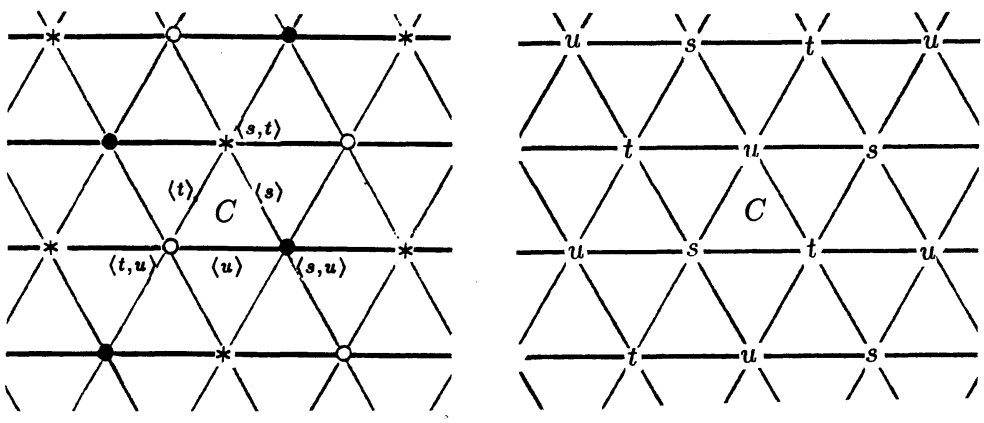
$$d : \mathcal{C} \times \mathcal{C} \to \mathbf{Z},$$

where $d(C, D)$ is the minimal length of a gallery from C to D. In this lecture we will take a close look at minimal galleries, and we will see that we can associate to a pair (C, D) much more than just the number $l = d(C, D)$. The upshot of our investigation will be a new way of axiomatizing buildings, which has become quite important in recent research. We need to begin by explaining how to associate a matrix to Δ which captures its "type", i.e., the isomorphism type of the apartments.

1. The Coxeter matrix of a building

Consider first the case where Δ is the Coxeter complex $\Sigma = \Sigma(W, S)$ associated to a Coxeter group (W, S). [See Appendix A for the notation.] Let $M = (m(s,t))_{s,t \in S}$ be the Coxeter matrix of (W, S); thus $m(s,t)$ is the order of st. This has a geometric interpretation which is explained in §5 of Appendix A, and which we review briefly here.

Recall that Σ is labellable. In fact, there is a canonical labelling, with S as the set of labels, which is defined in §3 of Appendix A. An example of this is illustrated below, where Σ is the plane tiled by equilateral triangles, and W is generated by reflections s, t, u with respect to the sides of one "fundamental" triangle. In the picture on the left, a labelling is indicated by the use of three "colors" for the vertices; in addition, the stabilizers of the faces of the fundamental chamber are shown. The picture on the right shows the canonical labelling, with s, t, u as labels.



Returning to the general case, every simplex $\sigma \in \Sigma$ has a well-defined *type*, which is a subset of S. In particular, a codimension 2 simplex σ has type $S - \{s,t\}$ for some $s, t \in S$ with $s \neq t$. The geometric interpretation of the matrix M, then, is that the link of σ is a $2m$-gon, where $m = m(s,t)$. (Note that m could be ∞, in which case an ∞-gon is to be interpreted as a line.) In the example above, for instance, all the numbers m are equal to 3, and the link of any vertex is a hexagon.

Note that a $2m$-gon has diameter m, so we can also write

$$m(s,t) = \operatorname{diam}(\operatorname{lk} \sigma).$$

One consequence of this is the following: Suppose we are given a Coxeter complex Σ as an abstract simplicial complex, but we are not told what Coxeter group W it comes from. Then we can reconstruct W from Σ by looking at the links of the various types of codimension 2 simplices, where "type" makes sense because Σ is labellable. Here is a convenient way to state this result:

Proposition 1. *Let Σ be a Coxeter complex, labelled by a set I. Then there is a well-defined matrix $M = (m_{ij})_{i,j \in I}$ with $m_{ii} = 1$ and, for $i \neq j$,*

$$m_{ij} = \operatorname{diam}(\operatorname{lk} \sigma)$$

for any simplex σ of type $I - \{i,j\}$. Let W_M be the Coxeter group defined by M, with generating set $S = \{s_i\}_{i \in I}$ and relations $(s_i s_j)^{m_{ij}} = 1$, and let Σ_M be the associated Coxeter complex $\Sigma(W_M, S)$. Then $\Sigma \approx \Sigma_M$.

Remark. There is some ambiguity in our use of the word "diameter" above. On the one hand, the diameter of a chamber complex is defined as the supremum of the gallery distances $d(C,D)$, where C and D range over all chambers. On the other hand, the links discussed above are graphs, for which one usually defines diameter in terms of lengths of edge paths between vertices. This ambiguity is harmless, since a $2m$-gon has diameter m in both senses.

It is now easy to generalize to buildings. Let Δ be an arbitrary building, with an arbitrary system of apartments. Recall that Δ is labellable (Lecture 1, §3, Proposition 6). Choose a fixed labelling by a set I. In view of the essential uniqueness of labellings (Appendix A, §3), nothing we do will depend on this choice in any significant way.

Proposition 2. *Let Δ be a labelled building as above. There is a well-defined matrix $M = (m_{ij})_{i,j\in I}$ with $m_{ii} = 1$ and, for $i \neq j$,*

$$m_{ij} = \operatorname{diam}(\operatorname{lk}\sigma)$$

for any simplex σ of type $I - \{i,j\}$. Moreover, every apartment Σ is isomorphic to Σ_M.

This is a fairly easy consequence of Proposition 1, once one checks two things: (a) For any simplex $\sigma \in \Delta$, its link $\operatorname{lk}_\Delta \sigma$ is a building with apartments $\operatorname{lk}_\Sigma \sigma$, where Σ ranges over the apartments of Δ. (b) Any building has the same diameter as its apartments. The proof of (a) is easy, right from the definitions. And (b) follows from the convexity of apartments (Lecture 1, §3, Proposition 5). For more details see [10], §§IV.1 and IV.3.

The reader might find it an instructive exercise to verify the conclusions of Proposition 2 directly in case $\Delta = \Delta(k^n)$.

Remark. Proposition 2 shows that the isomorphism type of the apartments depends only on Δ; this was stated without proof in Lecture 1 (§3, Proposition 2).

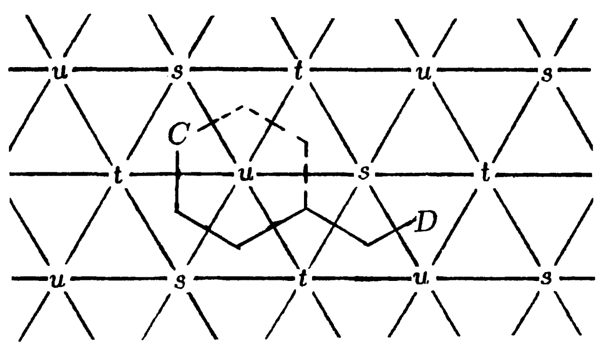
The matrix M is called the *Coxeter matrix* of Δ, and the associated Coxeter group $W = W_M$ is called the *Weyl group* of Δ.

2. Galleries and words

We continue with the notation just established: Δ is a labelled building, M is its Coxeter matrix, and W is its Weyl group, with distinguished generating set S in 1-1 correspondence with the set I of labels. It is convenient to identify I with S and thereby to regard S as the set of labels. This double use of S (as a set of group elements and a set of labels) is potentially confusing, but it turns out to be quite convenient.

We now focus on the set $\mathcal{C}$ of chambers of Δ. For the moment, we view $\mathcal{C}$ as a set with a relation (adjacency). Recall that this relation enabled us to define the gallery metric d, making $\mathcal{C}$ a metric space. Using the labelling, we will refine the adjacency relation and obtain from this a refined distance function.

Given a label $s \in S$, we say that two chambers C, D are *s-adjacent*, and we write $C \overset{s}{\sim} D$, if C and D have the same face of type $S - \{s\}$. Note, then, that any two distinct adjacent chambers are s-adjacent for a unique $s \in S$. Consequently, a non-stuttering gallery $C_0, \ldots, C_l$ has a well-defined *type* $\mathbf{s} = (s_1, \ldots, s_l)$, such that $C_{i-1} \overset{s_i}{\sim} C_i$ for $i = 1, \ldots, l$. For example, the solid line in the picture below indicates a gallery of type (s, t, s, u, t) between two chambers C and D. If, on the other hand, we follow the broken line instead of the solid line at the beginning, then we obtain a gallery of type (t, s, t, u, t) between the same two chambers. Similarly, there is a gallery of type (t, s, u, t, u) from C to D.



Switching now to the other role played by S (as a set of generators of W), note that the type $\mathbf{s}$ of a gallery can be viewed as a *word.* Such a word represents an element of W. For instance, the type (s, t, s, u, t) which arose in our example above represents the element $w = stsut \in W$. Note, in this example, that the other two galleries from C to D that we mentioned have types that represent the same element w. Indeed, the relations $(st)^3 = 1$ and $(tu)^3 = 1$ imply that

$$stsut = tstut = tsutu$$

in W. Thus we seem to have a well-defined element $w \in W$ associated to the ordered pair (C, D). This illustrates a general principle, valid in any building:

Theorem. *There is a unique function $\delta : \mathcal{C} \times \mathcal{C} \to W$ with the following property: For any $C, D \in \mathcal{C}$, if there is a minimal gallery from C to D in Δ of type $\mathbf{s} = (s_1, \ldots, s_l)$, then $\delta(C, D)$ is the element $w = s_1 \cdots s_l$ represented by* $\mathbf{s}$. *Moreover, $d(C, D) = l(\delta(C, D))$, where $l : W \to \mathbf{Z}$ is the length function l_S.*

Thus δ is the promised refinement of the distance function d:

$$\begin{tikzcd} & W \arrow[d, "l"] \\ \mathcal{C} \times \mathcal{C} \arrow[ur, "\delta"] \arrow[r, "d"] & \mathbf{Z} \end{tikzcd}$$

I like to think of $w = \delta(C, D)$ as something like a "vector" pointing from C to D. It has a "magnitude" $l(w)$, which is the distance from C to D, but it contains a great deal more information; for example, reduced decompositions of w are related to "geodesics" from C to D.

The proof of the theorem is not difficult. First, by the convexity of apartments, every minimal gallery between two given chambers is contained in any apartment containing those chambers. So we reduce easily to the case where Δ is a single apartment, which may be assumed to be $\Sigma(W, S)$ with its canonical labelling. The proof in this case is then an easy consequence of the well-known correspondence between galleries in $\Sigma(W, S)$ and S-words; see Appendix A, §6, for more details. Incidentally, that proof also shows that δ in this special case can be identified with the "difference function"

$$W \times W \to W,$$

given by $(w, w') \mapsto w^{-1}w'$. Thus the last assertion of the theorem reduces to the familiar formula $d(w, w') = l(w^{-1}w')$ for the word metric.

3. A new axiomatization of buildings

We have just seen that a building gives rise to a pair $(\mathcal{C}, \delta)$ consisting of a set with a W-valued distance function. It is not hard to see that this assignment is essentially 1-1, i.e., that one can recover Δ up to canonical isomorphism from $(\mathcal{C}, \delta)$. (The point here is that we can recover the s-adjacency relations from δ, since $C \overset{s}{\sim} D \iff \delta(C, D) \in \{1, s\}$; it is then easy to reconstruct Δ, cf. §D of the appendix to Chapter I of [10].) The obvious question, then, is which pairs $(\mathcal{C}, \delta)$ arise from buildings? In other words, what are the appropriate axioms for "W-metric spaces", in order that they correspond precisely to buildings?

Tits [27] has recently given the following beautiful answer to this question:

Theorem. *Let (W, S) be a Coxeter system with S finite. Given a set $\mathcal{C}$ and a function $\delta : \mathcal{C} \times \mathcal{C} \to W$, the pair $(\mathcal{C}, \delta)$ arises from a building if and only if it satisfies the following four axioms:*

(1) *$\delta(C, D) = 1$ if and only if $C = D$.*
(2) *$\delta(D, C) = \delta(C, D)^{-1}$.*
(3) *If $\delta(C', C) = s \in S$ and $\delta(C, D) = w$, then $\delta(C', D) = sw$ or w. If, in addition, $l(sw) = l(w) + 1$, then $\delta(C', D) = sw$.*
(4) *If $\delta(C, D) = w$, then for any $s \in S$ there is a C' such that $\delta(C', C) = s$ and $\delta(C', D) = sw$. If $l(sw) = l(w) - 1$, then there is a unique such C'.*

Remarks. 1. The first three axioms resemble the three axioms for (ordinary) metric spaces. Axiom (3), for example, or at least the first part of (3), is something like the triangle inequality. The second part of (3), from this point of view, gives a "collinearity" condition under which equality holds in the triangle inequality. Axiom (4), on the other hand, is probably harder to grasp intuitively; our discussion of the tree case below should help.

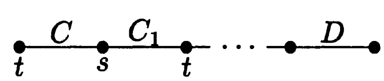
2. One can actually get by with a simpler set of axioms. For example, it is not hard to show that (1), (3), and the first part of (4) imply (2) and the second part of (4).

3. Perhaps the most interesting aspect of the theorem is that it provides an axiomatization of buildings that does not assume the existence of anything resembling apartments. The heart of the proof of the theorem is the construction of apartments, in the guise of "strong isometries" from W into $\mathcal{C}$. A *strong isometry* is a function that preserves the W-valued distance function, where W is equipped with the distance function $(w, w') \mapsto w^{-1}w'$ as in §2 above.

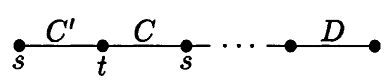
See Appendix B for a sketch of the proof of the theorem. See also Tits [25] for an earlier axiomatization in the same spirit.

We close this lecture by explaining why (3) and (4) are true in the tree case. Assume, then, that Δ is a tree (in which every vertex has degree ≥ 2). The Weyl group W in this case is the infinite dihedral group, and S consists of two generators of order 2. If one carefully checks the definition of the adjacency relations, one finds that two adjacent edges are s-adjacent ($s \in S$) if and only if their common vertex does *not* have label s.

Now fix two chambers C, D, and let $\delta(C, D) = w$. To avoid uninteresting cases, assume $w \neq 1$. Suppose $\delta(C', C) = s \in S$, and let t be the other element of S. Thus the common vertex of C and C' has label t. Assume first that $l(sw) = l(w) + 1$, i.e., that the (unique) S-word representing w starts with t. Let $\Gamma = (C_0, C_1, \ldots, C_l)$ be the (unique) minimal gallery from C to D. Then we have $C = C_0 \overset{t}{\sim} C_1$, so C and C_1 have a common vertex with label s. We can therefore picture Γ as the following edgepath:

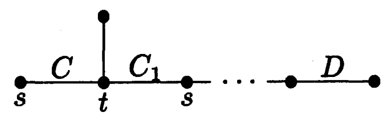


Putting C' in front of Γ, we obtain the path



This path is a geodesic since $C' \neq C$. The composite gallery $(C', C_0, C_1, \ldots, C_l)$ is therefore minimal, whence $\delta(C', D) = sw$. This proves (3) and the first part of (4) when $l(sw) = l(w) + 1$.

Suppose now that $l(sw) = l(w) - 1$, i.e., that w starts with s. The labels on Γ are then reversed from those above, but C and C' still share a vertex with label t. The picture therefore takes the form



where the vertical edge is C'. Now it is possible that $C' = C_1$; in this case we have the minimal gallery $C' = C_1, \ldots, D$ whose type is obtained from that of Γ by removing the initial s. Hence $\delta(C', D) = sw$ for this particular C'. For all other choices of C', we have a minimal gallery $C', C_1, \ldots, D$ of the same type as Γ, whence $\delta(C', D) = w$. All assertions in (3) and (4) should now be clear.

Lecture 3. Buildings and Groups

Recall that every Coxeter system (W, S) has an associated simplicial complex $\Sigma = \Sigma(W, S)$, on which W acts as a group of type-preserving simplicial automorphisms. [In fact, one can show that W is the full group of type-preserving simplicial automorphisms of Σ.] The complexes Σ which correspond to Coxeter groups in this way are precisely the thin buildings.

It is natural to try to generalize this construction. Thus we seek a class of groups G to which we can associate a building Δ. By analogy with the special case of Coxeter groups, we expect G to act on Δ by type-preserving automorphisms, and we expect the action to be transitive on chambers. One might also hope that the action would be transitive on apartments. It turns out that the most satisfactory theory is obtained by demanding a transitivity property which is strong enough to imply both of those just mentioned.

1. Strongly transitive group actions

Let Δ be a building and $\mathcal{A}$ a system of apartments. Let a group G act on Δ by type-preserving automorphisms leaving $\mathcal{A}$ invariant. The action is said to be *strongly transitive* if G acts transitively on the set of pairs (Σ, C) with $\Sigma \in \mathcal{A}$ and $C \in \operatorname{Ch}\Sigma$, where the latter is the set of chambers of Σ.

Note that the action is strongly transitive if and only if G is transitive on $\mathcal{A}$ and the stabilizer of some $\Sigma \in \mathcal{A}$ is transitive on $\operatorname{Ch}\Sigma$. Alternatively, the action is strongly transitive if and only if G is transitive on $\operatorname{Ch}\Delta$ and the stabilizer of some $C \in \operatorname{Ch}\Delta$ is transitive on the set of apartments containing C.

Our goal is to discover a class of groups G for which we can construct a building with a strongly transitive G-action. We will do this by working the problem backwards: We assume that we have a strongly transitive action, and we will see what structure this imposes on G. Fix an apartment $\Sigma \in \mathcal{A}$ (the "fundamental apartment") and a chamber $C \in \operatorname{Ch}\Sigma$ (the "fundamental chamber"). We need some notation for stabilizers, etc.:

Let B be the stabilizer of C; it acts transitively on the set of apartments containing C.

Let N be the stabilizer of Σ; it leaves Σ invariant and acts transitively on $\operatorname{Ch}\Sigma$. One can deduce that N surjects onto the group $\operatorname{Aut}_0 \Sigma$ of type-preserving automorphisms of Σ. [Given $\phi \in \operatorname{Aut}_0 \Sigma$, there is an $n \in N$ such that $nC = \phi(C)$. Then ϕ and n agree pointwise on C since they are both type-preserving; a standard argument based on the thinness of Σ now shows that ϕ and n agree on all of Σ. For more details, look at the "standard uniqueness argument" in [10].]

Let T be the fixer of Σ, i.e., the (normal) subgroup of N consisting of those elements that fix Σ pointwise. In other words,

$$T = \ker\{N \twoheadrightarrow \operatorname{Aut}_0 \Sigma\}.$$

Note that we can also describe T as $N \cap B$; this follows from the same standard uniqueness argument used above.

Let W be the quotient group N/T. It is isomorphic to $\operatorname{Aut}_0 \Sigma$, hence it is a Coxeter group whose associated Coxeter complex is isomorphic to Σ. More precisely, the action of W on Σ yields a set S of "fundamental reflections", these being the non-trivial elements of W which fix a codimension 1 face of the fundamental chamber C, and (W, S) is a Coxeter system such that $\Sigma(W, S)$ is canonically isomorphic to Σ. [The point here is that Σ is known to be a Coxeter complex, so the assertions just made about W are known to be true about the group $\operatorname{Aut}_0 \Sigma$, which is canonically isomorphic to W.]

The notation that has been introduced so far is summarized in the following diagram:

$$\begin{tikzcd}[column sep=small] & G \arrow[dl, dash] \arrow[dr, dash] & & \\ B \arrow[dr, dash] & & N \arrow[r, two heads] \arrow[dl, dash] & W = \langle S \rangle \\ & T & & \end{tikzcd}$$

We now give Σ its canonical labelling with S as the set of labels; this labelling is characterized by the property that $wC \overset{s}{\sim} wsC$ for all $w \in W$ and $s \in S$, where C is still the fundamental chamber. Extend this labelling to a labelling of Δ. Consideration of types of minimal galleries then yields, as in Lecture 2, a function

$$\delta : \operatorname{Ch}\Delta \times \operatorname{Ch}\Delta \to W.$$

Since the action of G is label-preserving, δ is G-invariant:

$$\delta(gC_1, gC_2) = \delta(C_1, C_2)$$

for all $C_1, C_2 \in \operatorname{Ch}\Delta$ and all $g \in G$.

Before proceeding further, let's look at an example.

2. Example

Let $\Delta = \Delta(k^n)$ as in Lecture 1 (§1, Example 1). Let G be the general linear group $\mathrm{GL}_n(k)$. It acts on k^n and permutes the subspaces, hence it acts on Δ. The action is easily seen to be type-preserving and strongly transitive. As fundamental chamber we take the "standard flag"

$$[e_1] < [e_1, e_2] < \cdots < [e_1, \ldots, e_{n-1}]$$

constructed from the standard basis vectors. And as fundamental apartment, we take the apartment determined by the standard basis vectors; its chambers are the flags

$$[e_{\pi(1)}] < [e_{\pi(1)}, e_{\pi(2)}] < \cdots < [e_{\pi(1)}, \ldots, e_{\pi(n-1)}],$$

where π ranges over the permutations of $\{1, \ldots, n\}$.

The stabilizer B of C is the group of upper triangular matrices, called the *Borel* subgroup of G; this explains the use of the letter "B" in the general theory. The stabilizer N of Σ is the monomial group, i.e., the group of matrices with exactly one non-zero entry in each row and in each column. The intersection $T = N \cap B$ is the group of diagonal matrices. And the quotient $W = N/T$ can be identified with the symmetric group on n letters. The letters "T", "N", and "W" are reminders that T is a maximal *torus* in G, N is the *normalizer* of T, and W is the *Weyl group*.

Finally, the reader might enjoy trying to guess what δ is in this example. It associates to any two maximal flags in k^n a certain permutation of $\{1, \ldots, n\}$. This turns out to be the so-called *Jordan–Hölder permutation*. See [10], §IV.2, Exercise 2; see also [1] and [2].

3. The structure of G

We return to the general setup of §1. In order to illustrate the ideas of Lecture 2, we will focus our attention on the set $\mathrm{Ch}\,\Delta$, together with the function $\delta : \mathrm{Ch}\,\Delta \times \mathrm{Ch}\,\Delta \to W$. We will identify $\mathrm{Ch}\,\Delta$ with the set G/B of left cosets of B in G, interpret δ group-theoretically, and then deduce results about G. See Chapter V of [10] for a different way of deriving the same results.

Recall first that δ is G-invariant, so that

$$\delta(gC, hC) = \delta(C, g^{-1}hC)$$

for any $g, h \in G$, where C, as always in this lecture, is the fundamental chamber. Thus δ is completely known as soon as we know $\delta(C, gC)$ for all g. Now strong transitivity implies, for any $g \in G$, that there is a $b \in B$ such that $bgC \in \Sigma$. [Choose an apartment containing C and gC, and use the transitivity of B on the set of apartments containing C.] Since W acts transitively on $\operatorname{Ch}\Sigma$, we can write $bgC = wC$ for some $w \in W$. Using the invariance of δ under the action of b, we now obtain

$$\begin{aligned}\delta(C, gC) &= \delta(C, bgC)\\ &= \delta(C, wC)\\ &= w.\end{aligned}$$

The last equation here follows from the correspondence between galleries in $\Sigma(W, S)$ and S-words, cf. Appendix A, §1.

We now use the fact that chambers are in 1-1 correspondence with cosets of B, so that the equation $bgC = wC$ above can also be written as $bgB = wB$. [Here wB denotes the coset nB for any representative $n \in N$ of w; this is independent of the choice of n.] Consequently, $gB = b^{-1}wB$, whence $g \in BwB$ ($= BnB$ for any representative n as above). Conversely, if $g \in BwB$ then $bgC = wC$ for some $b \in B$. The upshot of the previous paragraph, then, is that every $g \in G$ is in some double coset BwB, and that one then has $\delta(C, gC) = w$.

An immediate consequence of this is that g is in a *unique* double coset BwB. In other words:

Theorem 1. $G = BWB \overset{\text{def}}{=} \bigcup_{w\in W} BwB = \coprod_{w\in W} BwB.$

For historical reasons that will be explained in the next lecture, this result is called the *Bruhat decomposition* of G. Note that it implies, in particular, that G is generated by B and N. The reader might find it helpful at this point to think about what the Bruhat decomposition means when $G = \mathrm{GL}_n(k)$ as in §2; there is a concrete interpretation in terms of row and column operations.

Returning to the study of δ, let's now view δ as a function $G/B \times G/B \to W$. Our calculation above then says that $\delta(B, gB)$ is the unique $w \in W$ such that $BgB = BwB$. Since $\delta(gB, hB) = \delta(B, g^{-1}hB)$, we conclude that $\delta(gB, hB)$ is the element w which represents the double coset $Bg^{-1}hB = (gB)^{-1}(hB)$. In other words, we have arrived at a group-theoretic description of δ as the composite

$$G/B \times G/B \to B\backslash G/B \to W,$$

where the second arrow is given by the Bruhat decomposition and the first arrow is the "difference map" $(gB, hB) \mapsto (gB)^{-1}(hB)$. Viewed in this way, δ appears as a very natural generalization of the analogous function on W itself which arose in Lecture 2.

We close this section by giving the group-theoretic translation of properties (3) and (4) of δ stated in Lecture 2. This translation involves products of double cosets, about which one can ordinarily say practically nothing.

Theorem 2. *For any $s \in S$ and $w \in W$, we have:*

(a) *If $l(sw) = l(w) + 1$, then $BsB \cdot BwB = BswB$.*

(b) *If $l(sw) = l(w) - 1$, then $BsB \cdot BwB \subseteq BswB \cup BwB$. Equality holds if Δ is thick.*

Proof. Take $g \in BsB$ and $h \in BwB$. Then the double coset containing the product gh is what arises when one computes $\delta(g^{-1}B, hB)$. Since $\delta(g^{-1}B, B) = \delta(B, gB) = s$ and $\delta(B, hB) = w$, we can apply the "triangle inequality" (i.e., axiom (3)) to $g^{-1}B, B, hB$; we conclude that $\delta(g^{-1}B, hB) = sw$ or w, with the first case occurring if $l(sw) = l(w) + 1$. Hence $gh \in BswB$ or BwB, with the first case occurring if $l(sw) = l(w) + 1$. This proves (a) and the first part of (b).

Assume now that $l(sw) = l(w) - 1$, and fix $h \in BwB$. By axiom (4) there is a unique coset $g_0^{-1}B$ with $\delta(g_0^{-1}B, B) = s$ and $\delta(g_0^{-1}B, hB) = sw$; for all other cosets $g^{-1}B$ with $\delta(g^{-1}B, B) = s$, we have $\delta(g^{-1}B, hB) = w$. Now if Δ is thick, then there must in fact exist at least one such g. [In other words, the common face of $g_0^{-1}C$ and C is contained in at least one other chamber.] We then have $g \in BsB$ and $gh \in BwB$. Thus $BsB \cdot BwB$ meets BwB, and the second part of (b) follows easily. $\square$

Note that we can take $w = s$ in (b). If Δ is thick, we conclude that $BsB \cdot BsB \not\subseteq B$, or, equivalently, that $sBs \not\subseteq B$. Since s has order 2, we can also write this as

$$sBs^{-1} \not\subseteq B.$$

In conclusion, we have seen in this section that a strongly transitive action of a group on a building gives rise to a pair of subgroups B, N with some very special properties. We now reverse the procedure and show that a "BN-pair" in an abstract group is sufficient for the construction of a building. For simplicity, we will only consider the thick case, which suffices for most applications.

4. BN-pairs

We say that a pair of subgroups B and N of a group G is a *BN-pair* if B and N generate G, the intersection $T = B \cap N$ is normal in N, and the quotient $W = N/T$ admits a set of generators S such that the following two conditions hold for all $s \in S$ and $w \in W$:

(BN1) $$BsB \cdot BwB \subseteq BswB \cup BwB.$$

(BN2) $$sBs^{-1} \not\subseteq B.$$

One also says, in this situation, that the quadruple (G, B, N, S) is a *Tits system*.

The reader may be surprised at how little is assumed here, given how much more is known to hold in the setup of §3. For example, the definition does not require W to be a Coxeter group. It does not even require the elements of S to be of order 2. The reason for not assuming more is that everything one needs turns out to be a consequence of our two axioms. Indeed, one derives, by fairly short group-theoretic arguments, all of the following results about BN-pairs (cf. [10], §V.2):

(1) $G = \coprod_{w \in W} BwB$.

(2) $BsB \cdot BwB = \begin{cases} BswB & \text{if } l(sw) = l(w) + 1 \\ BswB \cup BwB & \text{if } l(sw) = l(w) - 1. \end{cases}$

(3) S consists of elements of order 2, and (W, S) is a Coxeter system.

(4) S is uniquely determined by B and N. [This explains why we define "BN-pairs" instead of "BNS-triples".]

(5) The subgroups P with $B \subseteq P \subseteq G$ are in 1-1 correspondence with the special subgroups $W' = \langle S' \rangle$ of W (where S' ranges over all subsets of S) via $W' \leftrightarrow BW'B$. Moreover, these subgroups P are mutually non-conjugate, and each is equal to its own normalizer.

The subgroups P in (5) were not mentioned explicitly in §3, but their significance in that context is that they are the stabilizers of the faces of the fundamental chamber C.

Finally, we state the result that we have been aiming for:

Theorem. *Given a BN-pair with S finite, one can construct a building Δ with G-action, together with a G-invariant system of apartments $\mathcal{A}$, with the following properties:*

(a) *The G-action is strongly transitive.*

(b) *There is a chamber C whose stabilizer is B.*

(c) *There is an apartment Σ containing C and stabilized by N.*

The proof is straightforward: One just directly constructs the simplices of the desired Δ by using cosets of the subgroups P discussed above (cf. [10], §V.3). Alternatively: Set $\mathcal{C} = G/B$ and define $\delta : \mathcal{C} \times \mathcal{C} \to W$ by letting $\delta(gB, hB)$ be the unique $w \in W$ such that $(gB)^{-1}(hB) = BwB$. It is not hard to verify that $(\mathcal{C}, \delta)$ satisfies the axioms stated in Lecture 2, and we therefore get a building [with G-action, since G acts on $(\mathcal{C}, \delta)$] by applying the main theorem of that lecture. Incidentally, we do not need to appeal to the hard part of the proof of the theorem just cited, which is the construction of apartments (in the form of strong isometries $W \to \mathcal{C}$). For we can use the canonical map

$$W = N/T \to G/B = \mathcal{C}$$

to get one apartment (the "fundamental" one), and we can get enough other apartments by applying the G-action.

Remark. N is not necessarily the full stabilizer of the fundamental apartment. In order to achieve this, one needs to add an extra axiom (called **(BN3)** in [10]) which says that N is "big enough".

There are two obvious questions that suggest themselves now. First, where do BN-pairs occur naturally? Second, what good is the building associated to a BN-pair, i.e., what can we discover about G once we have a building that G acts on? The remaining two lectures will be devoted to these two questions.

LECTURE 4. BN-PAIRS AND ALGEBRAIC GROUPS

We can best understand where BN-pairs arise "in nature" by looking at the history of the subject.

1. History

We begin with a paper of Bruhat [11] published in 1954. Bruhat was interested in representations of complex Lie groups, with emphasis on the classical matrix groups, such as $\mathrm{SL}_n(\mathbf{C})$. At the time of Bruhat's work, it had been known for a long time how to associate to such a group G a finite reflection group W, called the Weyl group of G. It is given by $W = N/T$, where T is a maximal "torus" and N is its normalizer. And people were becoming aware of the importance of a certain subgroup $B \subset G$, which eventually became known as the Borel subgroup of G as a result of the fundamental work of Borel [4]. What was not yet known, however, was the connection between B and W provided by the Bruhat decomposition $G = \coprod_{w \in W} BwB$.

Bruhat discovered this while studying so-called "induced representations". Questions about these led him to ask whether the set $B\backslash G/B$ of double cosets was finite. He was apparently surprised to discover, by a separate analysis for each of the families of classical groups, that the set of double cosets was not only finite but was in 1-1 correspondence with W.

Chevalley [13] immediately realized the importance of this result, and it became a basic tool in his work on algebraic matrix groups. He replaced Bruhat's case-by-case proof by a unified proof that applied not only to the classical groups but also to the five exceptional groups E_6, E_7, E_8, F_4, and G_2. Moreover, he worked over an arbitrary field, not just $\mathbf{C}$. In particular, his work included the construction of analogues of these exceptional groups over any field k. By letting k range over the finite fields, one obtained five new families of finite simple groups.

Meanwhile, Tits had been trying for some time to give geometric interpretations of the exceptional (complex) simple Lie groups. He thought E_6, for instance, should be the automorphism group of some sort of "geometry", in the same way that $\mathrm{SL}_n(\mathbf{C})$ is essentially the automorphism group of $(n-1)$-dimensional complex projective space. [More precisely, we should replace SL_n by the projective linear group here; and it is not really the full automorphism group of projective space, but it is a subgroup easily characterized in geometric terms.] By the time of the work of Bruhat and Chevalley, he had succeeded in doing this for some but not all of the exceptional groups G.

One motivation for Tits's project was that if one could describe G geometrically, then one should be able to construct analogues of G over an arbitrary field. This motivation disappeared as a result of Chevalley's work. But the question of finding geometric interpretations was still of intrinsic interest, and, after Chevalley, one could phrase that question as follows: Now that we know that the exceptional groups exist over any field, can we use the groups (or perhaps Chevalley's method of constructing them) to construct the geometries that we have been seeking?

So Tits studied Chevalley's methods, succeeded in constructing geometries for the "Chevalley groups", and extracted the axioms **(BN1)** and **(BN2)** as the essential properties that made this work. At more-or-less the same time, he wrote down axioms satisfied by his geometries, cf. [21]. These axioms are almost identical to the axioms for buildings that we stated in Lecture 1, except that they are stated in terms of incidence geometries instead of simplicial complexes. A reformulation of the axioms in terms of simplicial complexes appeared a few years later [22]. This reformulation was quite natural, since Tits had made extensive use of flags in [21], and these flags form the simplices of a simplicial complex.

[For the reader not familiar with this terminology, we remark that an incidence geometry involves "subspaces" of various dimensions, together with a relation called "incidence"; a flag is then a finite set of pairwise incident subspaces. For example, the building $\Delta(k^n)$ can be described as the flag complex of $(n-1)$-dimensional projective space.]

2. Algebraic groups and spherical buildings

It is clear now where we should expect to find BN-pairs. Namely, we should look at algebraic matrix groups. For example, we have already seen, in a somewhat roundabout way, how to construct a BN-pair in $G = \mathrm{GL}_n(k)$ for any field k. Our approach was to deduce this from the strongly transitive action of G on $\Delta(k^n)$; but, in fact, it is completely elementary to verify the BN-pair axioms by direct matrix computations. [It then follows that $\Delta(k^n)$ is in fact a building, which we stated without proof in Lecture 1; see [10], §V.5.] One can similarly treat $\mathrm{SL}_n(k)$, $\mathrm{Sp}_{2n}(k)$, and other matrix groups.

For the benefit of readers familiar with the language of algebraic groups, we state the general result: Every reductive algebraic group (over any field) gives rise to a BN-pair with finite Weyl group W, from which one obtains a spherical building (i.e., a building in which the apartments are finite and hence spheres). The result in this generality is due to Borel and Tits [7]; it is a vast generalization of the existence of BN-pairs in Chevalley groups.

Tits [23] has proven the remarkable fact that all thick spherical buildings of rank ≥ 3 (dimension ≥ 2) arise in this way. Thus we have a very good correspondence between spherical buildings and algebraic groups, except in dimension 1.

3. Algebraic groups and Euclidean buildings

It turns out that many of the same algebraic groups mentioned in §2 admit a *second* BN-pair, whenever the ground field comes equipped with a discrete valuation. This was first discovered by Iwahori and Matsumoto [16] for Chevalley groups and was greatly generalized by Bruhat and Tits [12]. The second BN-pair has an infinite Euclidean reflection group as its Weyl group, and the resulting building therefore has Euclidean spaces as apartments. Such a building is said to be of *Euclidean* (or *affine*) type. See §4 below for the precise definition. Tits [26] has shown that in rank ≥ 4 (dimension ≥ 3) every thick Euclidean building arises from an algebraic group over a field with discrete valuation.

As an example, we will describe the second BN-pair for $G = \mathrm{SL}_n(K)$, where K is a field with a discrete valuation $v : K \to \mathbf{Z} \cup \{\infty\}$. We will confine ourselves to a sketch; further details can be found in [10], §V.8.

Let A be the valuation ring associated to v, i.e.,

$$A = \{\, a \in K : v(a) \geq 0 \,\}.$$

It has a unique maximal ideal, which is the principal ideal πA, where $\pi \in A$ is any element with $v(\pi) = 1$. Let k be the residue field $A/\pi A$. Our point of view is that we are studying things (namely, matrix groups) defined over K, and we wish to "reduce" to a simpler field k as an aid in this study; a discrete valuation makes this possible by providing us with a nice ring A to serve as intermediary between K and k:

$$\begin{array}{ccc} A & \hookrightarrow & K \\ \downarrow & & \\ k & & \end{array}$$

In particular, we obtain our new B in G as the inverse image in $\mathrm{SL}_n(A)$ of the "ordinary B" in $\mathrm{SL}_n(k)$, i.e., the upper triangular subgroup of $\mathrm{SL}_n(k)$. On the other hand, we take N to be the same N we used before, namely, the monomial subgroup of G. [It would not make sense to also obtain N by lifting a subgroup of $\mathrm{SL}_n(k)$, since then B and N would both be subgroups of $\mathrm{SL}_n(A)$ and hence would not generate G.]

It is somewhat tedious (but not conceptually difficult) to verify that our new B and N do indeed form a BN-pair. To understand what type of building we get, we need to compute the Weyl group $W = N/T$.

Note first that $T = N \cap B$ is the group of monomial matrices in $\mathrm{SL}_n(A)$ which are upper triangular mod π. This is precisely the group $D(A)$ of diagonal matrices in $\mathrm{SL}_n(A)$. Now if we form $N/D(K)$ instead of $N/D(A)$ (where $D(K)$ is the diagonal subgroup of $G = \mathrm{SL}_n(K)$), then the quotient $\overline{W}$ is the "ordinary" Weyl group of G, i.e., the symmetric group on n letters. So there is a surjection $W \twoheadrightarrow \overline{W}$ whose kernel L is given by

$$L = D(K)/D(A) \approx (K^*/A^*)^{n-1} \approx \mathbf{Z}^{n-1}.$$

The asterisk here indicates the group of invertible elements of a ring, and the last isomorphism is induced by the valuation v. We therefore have an extension

$$1 \to L \to W \to \overline{W} \to 1.$$

It is not hard to show that the extension splits, so that

$$W \approx L \rtimes \overline{W}.$$

The action of $\overline{W}$ on L here can be described as follows: Let $V = \{ (x_1, \ldots, x_n) \in \mathbf{R}^n : \sum_{i=1}^n x_i = 0 \}$. Then $\overline{W}$ acts on V by permuting the coordinates, and we may identify L with the $\overline{W}$-invariant lattice $\mathbf{Z}^n \cap V$.

Using this description of W, one can see that it is a Euclidean reflection group acting on $V \approx \mathbf{R}^{n-1}$, with L acting as a lattice of translations. The corresponding Coxeter complex can then be identified with V itself (suitably triangulated). The building associated to the BN-pair is therefore a union of triangulated Euclidean spaces of dimension $n-1$. One can profitably think of it as an $(n-1)$-dimensional analogue of a tree. When $n = 2$, it really is a tree; in fact, it is the same as the tree described in Shalen's lectures [this volume] in terms of classes of lattices. One can give a similar description for arbitrary n.

Remarks. 1. We now have two buildings that $G = \mathrm{SL}_n(K)$ acts on. On the one hand, there is the spherical building $\Delta(K^n)$ that we get by forgetting that we have a valuation. Its apartments are spheres of dimension $n-2$. On the other hand, we have the Euclidean building just constructed, in which the apartments are Euclidean spaces of dimension $n-1$. Moreover, there is a 1-1 correspondence between the apartments in the first building and those in the second; for both sets of apartments are in 1-1 correspondence with G/N. [Recall that we used the same N in our two BN-pairs.] As we will see in Lecture 5, there is a geometric explanation for this: Roughly speaking, the spherical building is the "boundary" of the Euclidean building; it is obtained by adjoining a "sphere at infinity" to each apartment. Thus our two buildings fit together to form a single geometric object on which G acts.

2. There is a third building that comes to mind, namely, the (spherical) building $\Delta(k^n)$, on which $\mathrm{SL}_n(k)$ operates. This too shows up in our Euclidean building: It is isomorphic to the link of any vertex.

3. We have not assumed that the field K is complete with respect to the metric defined by the valuation v. What happens if K is incomplete and we pass to its completion $\hat{K}$? For example, we might have $K = \mathbf{Q}$ and $v = v_p$ (p-adic valuation) for some prime p, so that $\hat{K}$ is the field $\mathbf{Q}_p$ of p-adic numbers. The answer is that the Euclidean building associated to $\mathrm{SL}_n(\hat{K})$ is the *same* as the one associated to $\mathrm{SL}_n(K)$, but the apartment system is bigger. In fact, the apartment system that one gets from $\mathrm{SL}_n(\hat{K})$ is the complete one, i.e., the unique maximal one (which exists by Proposition 3′ of §3 of Lecture 1), whereas the apartment system that one gets from SL_n over an incomplete field is not (cf. [10], §VI.9F). Thus some of the geometry of Δ is hidden if K is incomplete.

We close this lecture by making some general remarks about Euclidean buildings, in preparation for the applications to be given in Lecture 5.

4. Introduction to Euclidean buildings

A general reference for this section is [10], Chapter VI.

By a *Euclidean reflection group* we will mean an essential, infinite, irreducible, affine reflection group W acting on Euclidean space $\mathbf{R}^n$ for some n. [Readers not familiar with this terminology will lose little by just thinking about a typical example, such as the group of isometries of the plane generated by the reflections with respect to the sides of an equilateral triangle.] The (affine) hyperplanes whose reflections are in W decompose $\mathbf{R}^n$ into chambers, which turn out to be simplices. Any choice of "fundamental" chamber C determines a set S of $n+1$ "fundamental" reflections, and there is then a canonical homeomorphism $|\Sigma(W,S)| \approx \mathbf{R}^n$. The vertical bars here indicate the geometric realization of an abstract simplicial complex.

The Euclidean reflection groups were classified by Witt: They are precisely the "affine Weyl groups" of the root systems associated to the simple complex Lie algebras, cf. [8]. Thus there is one Euclidean reflection group for each of the types $\mathrm{A}_n, \mathrm{B}_n, \ldots, \mathrm{G}_2$ in the usual list of simple Lie algebras. And it is these same affine Weyl groups that arise as the groups $W = N/T$ associated to the BN-pairs mentioned in §3 and illustrated with the example of $\mathrm{SL}_n(K)$. The buildings we are interested in, then, have apartments $\Sigma \approx \Sigma(W,S)$ for some (W,S) as in the previous paragraph. We will call such a building Δ a *Euclidean* building, although the more common term in the literature is *building of affine type*.

In discussing Euclidean buildings, we will shift point of view slightly and work with geometric realizations $|\Delta|$ instead of abstract simplicial complexes Δ. But it will be convenient to keep using the same terminology that we have been using up to now. Thus, for instance, we will now call $X = |\Delta|$ a building and we will call the subspaces $E = |\Sigma|$ $(\Sigma \in \mathcal{A})$ apartments. Similarly, a chamber will be viewed as a geometric (open) n-simplex $C \subset X$; the corresponding closed simplex (i.e., the topological closure of C in X) will be denoted $\overline{C}$.

It is not hard to show that each apartment E carries a canonical Euclidean metric d_E and that the isomorphisms $E \xrightarrow{\approx} E'$ that come from axiom **(B2)** are isometries. It follows that the metrics d_E extend to a well-defined function $d : X \times X \to \mathbf{R}$. The following proposition lays out the basic properties of this distance function.

Proposition.

(1) *d is a metric.*
(2) *X is complete with respect to d.*
(3) *For every apartment E and chamber $C \subset E$, the retraction $\rho = \rho_{E,C} : X \to E$ is distance-decreasing, i.e.,*

$$d(\rho(x), \rho(y)) \leq d(x, y).$$

Equality holds for $x \in \overline{C}$.
(4) *Any two points $x, y \in X$ are the endpoints of a unique geodesic segment (i.e., subset isometric to a closed interval of real numbers). It is contained in every apartment containing x and y, hence the apartments are convex.*
(5) *X is contractible and satisfies the CAT(0) inequality.*

Here CAT(0) is the comparison condition discussed in Paulin's lectures [this volume]; see also [15]. Roughly speaking, it says that X is a space of non-positive curvature.

We will sketch the proof of (1), (3), (4), and (5). All omitted details, as well as the proof of (2), can be found in [10], §VI.3, although the "CAT" terminology is not used there.

The first step is to prove (3), which makes sense even before we know that d is a metric. It follows from the definition of ρ (cf. Lecture 1) that ρ maps every apartment containing C isometrically onto E. This immediately yields the second assertion of (3), and it also shows that ρ maps every closed chamber isometrically onto its image. To prove the first assertion of (3), consider the line segment $[x, y]$ joining x and y in some apartment. One can subdivide $[x, y]$ so that each piece lies in a chamber; applying ρ, we get a polygonal path in E from $\rho(x)$ to $\rho(y)$, and the desired inequality now follows from the triangle inequality in E.

It is now easy to prove (1), the content of which is that d satisfies the triangle inequality: Given $x, y, z \in X$, choose an apartment E containing x and y, and let $\rho = \rho_{E,C}$ for some chamber C of E. Using the triangle inequality in E and the first part of (3), we find

$$d(x,y) \le d(x,\rho(z)) + d(\rho(z),y) \le d(x,z) + d(z,y),$$

as required.

We turn next to (4). The crucial observation is that the triangle inequality just proved is strict unless z is in the line segment $[x,y]_E$ joining x and y in the Euclidean space E. For suppose $d(x,y) = d(x,z)+d(z,y)$, and let z' be the unique point of $[x,y]_E$ with $d(x,z') = d(x,z)$ and $d(z',y) = d(z,y)$. Using the chain of inequalities above (which must be equalities under our present hypothesis), one easily concludes that $\rho(z) = z'$. Recall now that ρ was defined with respect to an arbitrary chamber C in E; in particular, we could have taken C with $z' \in \overline{C}$. The second assertion of (3) now implies that $z = z'$, which proves our claim that $z \in [x,y]_E$. We now know that

$$[x,y]_E = \{\, z \in X : d(x,y) = d(x,z) + d(z,y)\,\},$$

whence (4).

Finally, we prove CAT(0). (Contractibility is a formal consequence of this; it is also proven directly in [10], §VI.3.) A formulation of CAT(0) which will be convenient for us is the following: Given $w,x,y \in X$ and $t \in [0,1]$, let $z = (1-t)x+ty \in [x,y]$, the latter being the unique geodesic segment joining x and y; then

$$d^2(w,z) \le (1-t)d^2(w,x) + td^2(w,y) - t(1-t)d^2(x,y). \qquad (*)$$

To see why this is equivalent to CAT(0), consider the geodesic triangle with vertices w,x,y, and choose a comparison triangle in the Euclidean plane with vertices $\bar{w},\bar{x},\bar{y}$. Let $\bar{z} = (1-t)\bar{x} + t\bar{y} \in [\bar{x},\bar{y}]$. According to criterion C for CAT(0), one has to show $d(w,z) \le d(\bar{w},\bar{z})$, cf. Chapter 3 of [15]. Now one can compute $d(\bar{w},\bar{z})$ by Euclidean geometry, and one finds that its square is precisely the right-hand side of $(*)$. Hence $(*)$ is indeed equivalent to CAT(0).

We now prove $(*)$. Choose an apartment E containing x and y, and let C be a chamber of E with $z \in \overline{C}$. Let ρ be the retraction $\rho_{E,C} : X \to E$, and consider the triangle in E with vertices $\rho(w),x,y$. Using (3), and applying the formula from Euclidean geometry mentioned in the previous paragraph, we find

$$\begin{aligned} d^2(w,z) &= d^2(\rho(w),z) \\ &= (1-t)d^2(\rho(w),x) + td^2(\rho(w),y) - t(1-t)d^2(x,y) \\ &\le (1-t)d^2(w,x) + td^2(w,y) - t(1-t)d^2(x,y), \end{aligned}$$

as required.

We now have some idea what sorts of groups have BN-pairs and hence act on buildings. Our next goal is to see what this is good for.

A good example to keep in mind is $\mathrm{SL}_n(\mathbf{Q}_p)$ and its associated Euclidean building X of dimension $n-1$. Recall that X is a complete geodesic metric space satisfying CAT(0). We can get immediate applications of this by using a suitable fixed-point theorem.

1. The Bruhat-Tits fixed-point theorem and applications

Recall that a finite group acting on a tree always has a fixed point (cf. [20], §I.4.3, or Shalen's lectures in this volume). One also has the classical theorem of E. Cartan that a compact group of isometries of a complete simply-connected Riemannian manifold of non-positive curvature always has a fixed point. The following result of Bruhat and Tits [12] simultaneously generalizes these two theorems.

Theorem. *Let X be a complete geodesic metric space satisfying CAT(0). If G is a group of isometries of X with a bounded orbit, then G has a fixed point.*

The idea of the proof is extremely simple: One associates to every non-empty bounded set $A \subset X$ a "center" $c(A)$. The construction depends only on the metric on X and hence is compatible with isometries. Consequently, we can take A to be any non-empty G-invariant bounded set, such as a bounded orbit, and $c(A)$ is then fixed by G.

It remains to say how $c(A)$ is defined. The original definition given in [12] was somewhat awkward, but Serre later observed that the classical notion of "circumcenter" would do the job, i.e., that one could take $c(A)$ to be the center of the smallest closed ball containing A. More precisely, let $B_r(x)$ be the closed ball of radius r centered at x, and define the *circumradius* of A, denoted $r(A)$, to be the infimum of the numbers r such that $A \subseteq B_r(x)$ for some $x \in X$. Call x a *circumcenter* of A if this infimum is achieved at x, i.e., if $A \subseteq B_r(x)$ with $r = r(A)$. The claim is that A admits a unique circumcenter, which we can then take as our definition of $c(A)$.

This result turns out to be an easy consequence of the inequality $(*)$ that we used in Lecture 4 as our formulation of CAT(0). To prove uniqueness, for instance, suppose x and y are two distinct circumcenters. Then

$$A \subseteq B_r(x) \cap B_r(y),$$

where $r = r(A)$. On the other hand, if we set $z = (1-t)x + ty$ for any t with $0 < t < 1$, then there is an $s < r$ such that

$$B_r(x) \cap B_r(y) \subseteq B_s(z);$$

for we can apply $(*)$ with $w \in B_r(x) \cap B_r(y)$ to obtain

$$d^2(w,z) \leq r^2 - t(1-t)d^2(x,y),$$

so our assertion holds with $s = \sqrt{r^2 - t(1-t)d^2(x,y)}$. Thus $A \subseteq B_s(z)$, contradicting the minimality of r. This proves that there cannot be two distinct circumcenters.

Similar ideas are used to prove existence of the circumcenter. One shows that two "approximate circumcenters" x and y must be close together, so that a sequence of better and better approximations is a Cauchy sequence and hence converges to a circumcenter. Details can be found in [10], §VI.4.

As a sample application, we indicate how one can analyze the maximal compact subgroups of a group like $G = \mathrm{SL}_n(\mathbf{Q}_p)$. [This is a p-adic Lie group. What we are about to do is analogous to a classical application of the Cartan fixed-point theorem, in which one uses the latter to prove that a real Lie group has a unique conjugacy class of maximal compact subgroups.] Let X be the Euclidean building on which G acts. The fundamental chamber C has n vertices, whose stabilizers can be computed explicitly. One of them turns out to be $\mathrm{SL}_n(\mathbf{Z}_p)$, where $\mathbf{Z}_p$ is the ring of p-adic integers, i.e., it is the stabilizer in G of the standard lattice $\mathbf{Z}_p^n \subset \mathbf{Q}_p^n$. The other $n-1$ are stabilizers of other $\mathbf{Z}_p$-lattices.

All n of these subgroups are compact. In fact, they are maximal compact subgroups, since they are even maximal proper subgroups; moreover, they are mutually non-conjugate. [The general situation when one has a BN-pair is that the stabilizers of C and its non-empty faces are the proper subgroups of G containing B. No two of these are conjugate to one another, and the maximal ones are the stabilizers of the minimal faces of C, i.e., the vertex stabilizers. These are therefore maximal proper subgroups.] The Bruhat-Tits fixed-point theorem allows us to say more:

Corollary 1. *G has exactly n conjugacy classes of maximal compact subgroups, represented by the stabilizers of the vertices of C.*

Proof. In view of what has been said above, it suffices to prove that every compact subgroup $H \subset G$ fixes a vertex of X. Now the fixed-point theorem says that H fixes a point of X, hence H stabilizes a simplex of X; but then H fixes the vertices of that simplex, because the H-action is type-preserving. □

Similarly, one can use the fixed-point theorem to say things about torsion in discrete subgroups of G. For example:

Corollary 2. *If Γ is a discrete co-compact subgroup of G, then Γ has only finitely many conjugacy classes of finite subgroups.*

Sketch of proof. Γ acts on X with finite stabilizers and compact quotient. Let D be a compact subset of X that meets every Γ-orbit. Then Γ has only finitely many distinct stabilizers in D, and the fixed-point theorem implies that every finite subgroup of Γ is conjugate to a subgroup of one of them. □

Readers familiar with the cohomology theory of discrete groups will note that the action of Γ on X also leads to cohomological information. For example, Γ has virtual cohomological dimension $\leq n-1$, and its homology and cohomology groups are finitely generated.

As our next application of the theory of buildings, we would like to indicate how to get much sharper cohomological results. For this we need to know more about Euclidean buildings.

2. Some facts about Euclidean buildings

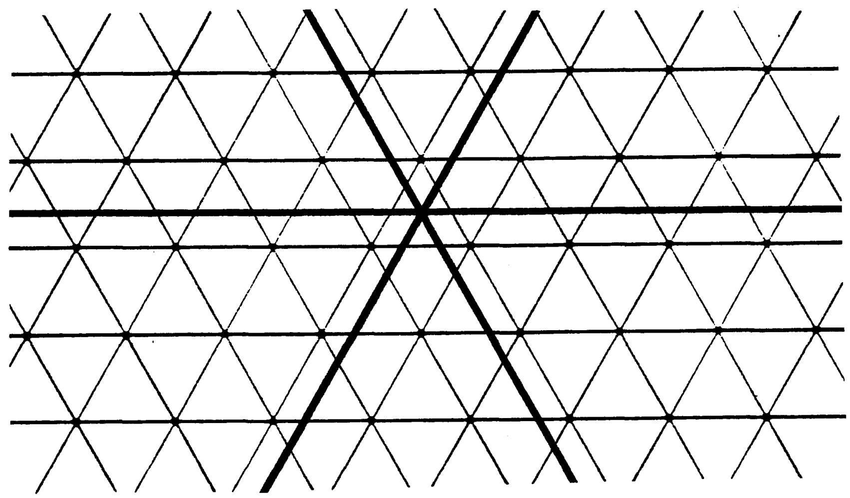
Throughout this section, X denotes an arbitrary Euclidean building, and d is its dimension. For example, X could be the building associated to $\mathrm{SL}_n(\mathbf{Q}_p)$, in which case $d = n-1$. In an effort to give the reader some feeling for the nature of Euclidean buildings, I will state more in this section than is strictly necessary for the application to group cohomology.

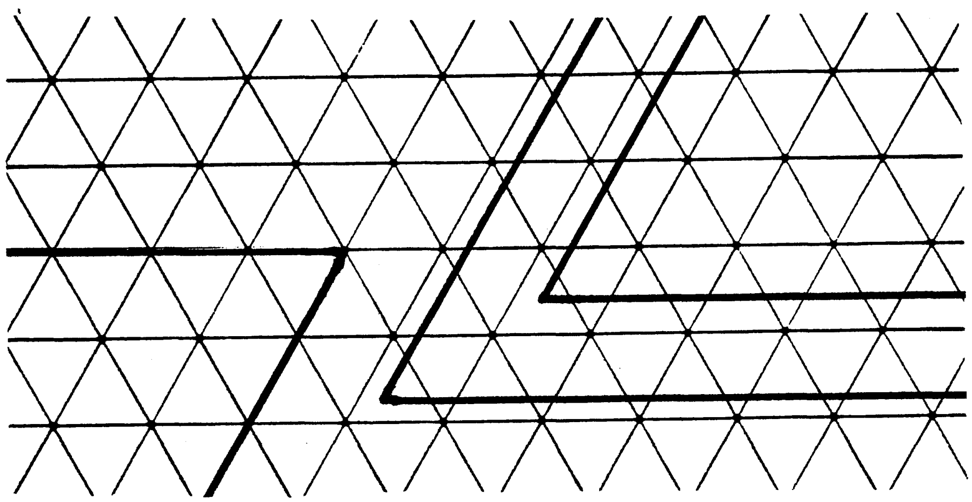
Theorem 1. *The complete apartment system consists of all subsets of X which are isometric to $\mathbf{R}^d$.*

The content of this statement is that every such subset is the geometric realization of a subcomplex, and that the family $\mathcal{A}$ of these subcomplexes is a system of apartments. One can get a feeling for this by thinking about the tree case, where the result is easy. One should also compare Theorem 1 to a combinatorial fact about arbitrary buildings, hinted at in §3 of Lecture 2 and essentially proved in Appendix B: If we view a building as a set $\mathcal{C}$ with a W-valued distance function, then the complete apartment system consists of all subsets of $\mathcal{C}$ strongly isometric to W. The proof of Theorem 1 is actually quite similar to the proof of this combinatorial fact; see [10], §VI.7.

The next result requires the notion of "sector". Let E be an apartment in X. Thus E is a Euclidean space divided up into simplicial chambers by walls, the latter being the reflecting hyperplanes associated to a Euclidean reflection group W. Associated to W is a canonical finite quotient $\overline{W}$, which is a finite reflection group. We can visualize it by choosing an origin in E and translating all the W-walls so that they pass through the origin; these translated hyperplanes are the $\overline{W}$-walls. Suppose, for example, that W is the group generated by the reflections with respect to the sides of an equilateral triangle. The apartment E is shown in the first picture below, with the three heavy lines being the $\overline{W}$-walls relative to one particular choice of origin. In this case $\overline{W}$ is the dihedral group of order 6.

The $\overline{W}$-walls divide E up into chambers which are simplicial cones. Each of these chambers is called a *sector* in E. More generally, in order to free the definition from our arbitrary choice of origin, we will also call any translate of a $\overline{W}$-chamber a sector. Finally, a subset of X is called a *sector* if it is a sector in some apartment (in the complete apartment system). The second picture shows some sectors. Note that the cone point of a sector need not be at a vertex. Note also that one can translate a sector into itself, thereby obtaining a *subsector*.





We can now state:

Theorem 2. *Given any two sectors $\mathfrak{C}, \mathfrak{D} \subset X$, there are subsectors $\mathfrak{C}' \subseteq \mathfrak{C}$ and $\mathfrak{D}' \subseteq \mathfrak{D}$ such that $\mathfrak{C}'$ and $\mathfrak{D}'$ are contained in some apartment.*

In the tree case, for instance, a sector is a ray, or half-line, and an apartment is a line. The reader should draw some pictures to see why any two rays have subrays which are contained in some line. The proof of Theorem 2 in the general case is more difficult and is given in [10], §VI.8.

We turn next to the ***boundary*** ∂X of X, or the set of points "at infinity". One can construct such a boundary for fairly general CAT(0) spaces, and the reader may be familiar with one or more special cases. The boundary of a tree, for instance, is its set of ends. And the boundary of a complete simply-connected Riemannian manifold of non-positive curvature is well-known in differential geometry, cf. [3] or [14]. In the present case, where X is a Euclidean building, the theory goes roughly as follows (cf. [10], §VI.9):

A point $e \in \partial X$ is represented by a ray (subset isometric to $[0, \infty)$), with two rays representing the same boundary point e if and only if they are at finite Hausdorff distance from one another.

One can organize the boundary points into simplices. For example, each sector $\mathfrak{C} \subset X$ determines a $(d-1)$-simplex (or "chamber") in ∂X, which can be visualized as the face of $\mathfrak{C}$ at infinity. [Recall that $\mathfrak{C}$ is a d-dimensional simplicial cone. The rays in $\mathfrak{C}$ starting at the cone point are parametrized by a $(d-1)$-simplex.] And the boundary points represented by rays in a given apartment E form a sphere (the "sphere at infinity"), decomposed into simplices by the sectors in E and their faces. This decomposition makes the sphere a Coxeter complex; the corresponding Coxeter group is the finite reflection group $\overline{W}$ mentioned above. Note that Theorem 2 implies that any two chambers in ∂X are contained in one of these spheres at infinity. All of this suggests the following result (cf. [10], §VI.9B):

Theorem 3. *There is a $(d-1)$-dimensional spherical building whose geometric realization is in 1-1 correspondence with ∂X.*

In case X is the Euclidean building associated to $\mathrm{SL}_n(\mathbf{Q}_p)$, for instance, the spherical building in Theorem 3 is just the usual spherical building $\Delta(\mathbf{Q}_p^n)$ that one gets by forgetting that $\mathbf{Q}_p$ has a valuation. This is proved in [10], §VI.9F.

Up to now we have been working with ∂X just as a set, with no topology. But one can impose a useful topology on ∂X, at least if X is locally compact. [And X will in fact be locally compact if it comes from an algebraic group over a field K with discrete valuation, provided the residue field k is finite.] One way to describe this topology is to choose a basepoint $x \in X$ and identify ∂X with the set of rays starting at x; we can view this set of rays as a set of maps $[0, \infty) \to X$ and give it the topology of uniform convergence on compact subintervals. It is not hard to show that the topology is independent of the choice of basepoint; moreover, ∂X, with this topology, is compact (Ascoli's theorem).

Using similar ideas, we can compactify X by adjoining ∂X "at infinity" (still assuming X is locally compact). Namely, we identify $X \cup \partial X$ with a space of maps $[0, \infty) \to X$, with points of X corresponding to maps which are isometries on a compact subinterval and constant thereafter. For future reference, we note that the compact space $X \cup \partial X$ is contractible.

Remarks. 1. The construction just sketched is well-known to work for any complete locally compact CAT(0) space, but I am not aware of any reference where the details are given in this generality.

2. The 1-1 correspondence in Theorem 3 is generally not a homeomorphism. In the tree case, for instance, ∂X might be homeomorphic to the Cantor set, whereas the spherical building in Theorem 3 is 0-dimensional and hence has a discrete geometric realization.

The final result we wish to state makes use of the topology on ∂X. Recall the Solomon–Tits theorem (Lecture 1, §3, Proposition 7), which implies that the reduced homology of a spherical building is non-trivial only in the top dimension, where it is free abelian. Borel and Serre ([6], 2.6) prove the analogue of this for the reduced cohomology $\tilde{H}^*(\partial X)$ of the compact space ∂X. [Here H^* denotes some reasonable cohomology theory for compact spaces, such as Alexander–Spanier cohomology.] Borel and Serre assume that X is the Euclidean building associated to an algebraic group over a complete field with discrete valuation and finite residue field, and they prove:

Theorem 4. *$\tilde{H}^i(\partial X)$ is trivial for $i \neq d-1$ and is free abelian for $i = d-1$.*

Remark. It seems certain that Theorem 4 remains valid for an arbitrary locally compact Euclidean building, but I have not checked this.

Since $X \cup \partial X$ is compact and contractible, one can use the long exact cohomology sequence of the pair $(X \cup \partial X, \partial X)$ to restate Theorem 4 in terms of the cohomology of X with compact supports:

Corollary. *$H_c^i(X)$ is trivial for $i \neq d$ and free abelian for $i = d$.*

3. Applications to the cohomology of discrete groups

The purpose of this short section is to give the reader a brief glimpse at one possible application of the theory of buildings. For a more detailed survey of this and related applications, see Chapter VII of [10].

The most obvious application, given what we have just done in §2, concerns discrete subgroups of p-adic Lie groups like $\mathrm{SL}_n(\mathbf{Q}_p)$. Let $\Gamma \subset \mathrm{SL}_n(\mathbf{Q}_p)$ be discrete and co-compact. For simplicity, assume further that Γ is torsion-free. Then Γ acts freely and co-compactly on the $(n-1)$-dimensional Euclidean building associated to $\mathrm{SL}_n(\mathbf{Q}_p)$. One can now use standard cohomological arguments to deduce from the corollary above that Γ has cohomological dimension exactly $n-1$ and that Γ satisfies a homological duality condition analogous to Poincaré duality ("Bieri–Eckmann duality").

More surprisingly, Borel and Serre [6] have applied the results of §2 to certain subgroups which are neither discrete nor co-compact. We will illustrate this by discussing the group $\mathrm{SL}_n(\mathbf{Z}[1/p])$ and its torsion-free subgroups of finite index.

We can view the ring $\mathbf{Z}[1/p]$ as a subring of both $\mathbf{R}$ and $\mathbf{Q}_p$. It is non-discrete in both cases, but for opposite reasons: In $\mathbf{R}$, we have $p^n \to \infty$ as $n \to +\infty$, and $p^n \to 0$ as $n \to -\infty$; in $\mathbf{Q}_p$, it is the other way around. One deduces that $\mathbf{Z}[1/p]$ embeds as a discrete subring of $\mathbf{R} \times \mathbf{Q}_p$. Consequently, we can view $\mathrm{SL}_n(\mathbf{Z}[1/p])$ as a discrete subgroup of $\mathrm{SL}_n(\mathbf{R}) \times \mathrm{SL}_n(\mathbf{Q}_p)$.

Now each of these factors has an associated contractible space on which it acts properly. In the case of $\mathrm{SL}_n(\mathbf{R})$, it is the usual symmetric space $X_\infty = \mathrm{SL}_n(\mathbf{R})/\mathrm{SO}_n(\mathbf{R})$, which is a manifold. In the case of $\mathrm{SL}_n(\mathbf{Q}_p)$, it is the Euclidean building which we have been discussing and which we now denote by X_p. This yields a proper action of $\mathrm{SL}_n(\mathbf{R}) \times \mathrm{SL}_n(\mathbf{Q}_p)$ on the product $X = X_\infty \times X_p$, and hence a properly discontinuous action of the discrete subgroup $\mathrm{SL}_n(\mathbf{Z}[1/p])$ on X.

If $\Gamma \subset \mathrm{SL}_n(\mathbf{Z}[1/p])$ is a torsion-free subgroup of finite index, then the action of Γ on X is free. It turns out, however, that the quotient $\Gamma \backslash X$ is not compact, so further work needs to be done before we can hope to draw the usual sorts of cohomological consequences.

This further work is done in an earlier paper of Borel and Serre [5]. There they attach a boundary to X_∞, obtaining a manifold with boundary $\bar{X}_\infty$. The construction is canonical enough that the action of Γ extends to the boundary. Moreover, if we replace X_∞ by $\bar{X}_\infty$ in the product above, then the action of Γ is still properly discontinuous, but now it is also co-compact.

The next step is to compute $H_c^*(\bar{X}_\infty \times X_p)$, for which it remains only to compute $H_c^*(\bar{X}_\infty)$. This computation, in turn, is reduced by standard duality theorems of algebraic topology to the computation of the homology of the boundary of $\bar{X}_\infty$. Here we find another application of buildings: Borel and Serre [5] show that the boundary they have attached to X_∞ has the homotopy type of the Tits building $\Delta(\mathbf{Q}^n)$ associated to $\mathrm{SL}_n(\mathbf{Q})$. Hence its reduced homology is non-trivial in only one dimension, where it is free abelian.

One can now deduce that Γ has finitely generated homology and satisfies Bieri–Eckmann duality; moreover, one gets an explicit calculation of its cohomological dimension.

4. Final remarks

In this lecture I have not even been able to hint at the range of applications of buildings. Many more applications are discussed in Tits's Vancouver lecture [24]. See also [17] and [19] for recent developments, with emphasis on connections with finite geometries and finite group theory.

Appendix A. Review of Coxeter Complexes

Most of what is summarized here can be found in P. de la Harpe's lectures [this volume] and/or in Chapters I–III of [10].

1. The complex associated to a Coxeter group

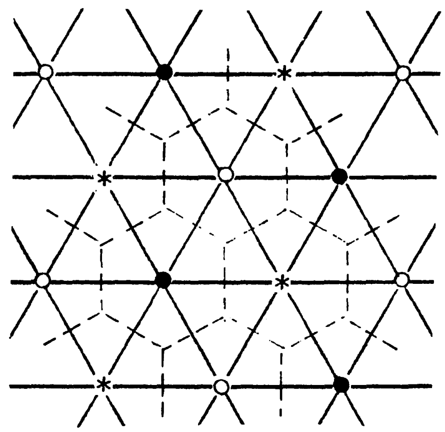
Let (W, S) be a Coxeter system of rank n (i.e., S has n elements). Recall that there is an associated simplicial complex $\Sigma = \Sigma(W, S)$ of rank n (dimension $n-1$), characterized by the following properties:

(1) W acts on Σ by simplicial automorphisms.
(2) There is a distinguished simplex C which is a fundamental domain for the action, in the following sense: Every simplex of Σ is W-equivalent to a unique face of C.
(3) The stabilizers of the faces of C (including C itself and the empty face) are the special subgroups of W, i.e., the subgroups generated by subsets of S, including S itself and the empty set.

In particular, the stabilizer of C is the trivial subgroup, so that W acts simply-transitively on the chambers (maximal simplices) of Σ. And the stabilizers of the codimension 1 faces of C are the subgroups $\langle s \rangle$ of order 2 generated by the elements of S. It follows that the set $\mathcal{C}$ of chambers of Σ can be identified with W (via $wC \leftrightarrow w$), and that two distinct elements $w, w' \in W$ correspond to adjacent chambers if and only if $w' = ws$ for some $s \in S$. [Another way to say this is that chambers correspond to vertices of the Cayley graph of (W, S), and pairs of distinct adjacent chambers correspond to edges of the Cayley graph.]

Recall that a *gallery* is a sequence of chambers $C_0, C_1, \ldots, C_l$ such that C_{i-1} and C_i are adjacent for $i = 1, \ldots, l$. It is said to *stutter* if $C_{i-1} = C_i$ for some i. In view of the previous paragraph, non-stuttering galleries in Σ starting at C are in 1-1 correspondence with S-words (and hence with paths starting at 1 in the Cayley graph). Under this correspondence, minimal galleries correspond to reduced words (and hence to geodesics in the Cayley graph). It follows that the metric spaces $\mathcal{C}$ and W are isometric, where the former has the gallery metric and the latter has the word metric.

The picture below illustrates the case where W is the group generated by the reflections in the sides of an equilateral triangle. Thus Σ is the Euclidean plane, tiled by equilateral triangles. The Cayley graph has been superimposed on Σ; it is the 1-skeleton of the dual tiling. Note also that the vertices of Σ have been drawn in three different "colors", to indicate the three W-orbits (or, in the language to be introduced in §3, the three *types* of vertices). The reader is advised to choose some words at random and to trace out the corresponding galleries in Σ. [One first has to choose a fundamental chamber in Σ or, equivalently, an origin in the Cayley graph.]



2. Coxeter complexes

By a *Coxeter complex* we will mean an abstract simplicial complex Σ which is isomorphic to $\Sigma(W,S)$ for some Coxeter system (W,S) (with S finite).

Examples. (a) The only 0-dimensional Coxeter complex is the 0-sphere S^0.

(b) The 1-dimensional Coxeter complexes are the $2m$-gons, $2 \le m \le \infty$.

(c) If X is the boundary of a regular convex polytope in $\mathbf{R}^n$, then the barycentric subdivision of X is a Coxeter complex; its simplices are the chains $F_0 < \cdots < F_q$ of non-empty faces of X.

(d) For any triple $\{p,q,r\}$ of integers ≥ 2, there is a well-known tiling of the "plane" (spherical, Euclidean, or hyperbolic, depending on the sign of $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1$) by triangles with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{r}$. The abstract simplicial complex underlying this tiling is a Coxeter complex. The case $p = q = r = 3$ was drawn above.

(e) If Σ is a Coxeter complex and $\sigma \in \Sigma$ is a simplex of codimension i, then the link of σ in Σ, denoted $\operatorname{lk}\sigma$ or $\operatorname{lk}_\Sigma \sigma$, is a Coxeter complex of rank i (dimension $i-1$). This assertion will be used later, so we explain it briefly: We may assume that $\Sigma = \Sigma(W,S)$ and that σ is a face of the fundamental chamber C. In this case the stabilizer of σ is a special subgroup $W' = \langle S' \rangle$, and I claim that $\operatorname{lk}\sigma \approx \Sigma(W',S')$. To prove the claim, one need only observe that W' acts on $\operatorname{lk}\sigma$ with a simplex σ' as fundamental domain and with the special subgroups of W' as the stabilizers of the faces of σ'; this is easy to check. Alternatively, one can deduce the claim from the definition of $\Sigma(W,S)$ in terms of "special cosets"; see [10], §III.2.

(f) If Σ_1 and Σ_2 are Coxeter complexes, then so is their join $\Sigma_1 * \Sigma_2$.

Remarks. 1. Tits has characterized Coxeter complexes in terms of the existence of enough "half-spaces"; see [10], §III.4.

2. Note that the Coxeter complexes in Examples (a)–(d) above are topologically either spheres or Euclidean spaces. The general fact is that every finite Coxeter complex is homeomorphic to a sphere and that every infinite Coxeter complex is contractible (but not necessarily homeomorphic to Euclidean space). See [10], §IV.6.

3. Labellings

Let Σ be a chamber complex, i.e., a finite-dimensional simplicial complex in which all the maximal simplices (called *chambers*) have the same dimension, and any two chambers can be connected by a gallery. Let n be the rank of Σ and let I be a set of cardinality n. A *labelling* of Σ by I is a function $\lambda : V(\Sigma) \to I$ (where $V(\Sigma)$ is the set of vertices of Σ), such that the vertices of any chamber are mapped bijectively onto I. If Σ can be labelled, then the labelling is essentially unique: Any two labellings (say by sets I and I') differ by a bijection $I \approx I'$. (To see this, just note that if the labelling is known on a chamber C, then it is determined on any chamber adjacent to C.)

It is sometimes helpful to think of a labelling as a "coloring" of the vertices. The number of colors used is required to be the rank of Σ, and joinable vertices are required to have different colors.

We often call $\lambda(v)$ the *type* of v. More generally, the *type* of a simplex σ is the subset of I consisting of the labels $\lambda(v)$ of the vertices of σ. This notion of "type" depends on λ, but the induced equivalence relation (where two simplices are equivalent if they have the same type) is independent of λ.

Not every chamber complex is labellable. (For example, a 1-dimensional complex is labellable if and only if it contains no closed polygon with an odd number of sides.) But Coxeter complexes are always labellable. To see this, we may assume $\Sigma = \Sigma(W, S)$; then the W-action partitions the vertices into n orbits, and we can label Σ by associating one label $i \in I$ to each orbit. The *canonical labelling* of $\Sigma(W, S)$ is obtained by taking $I = S$ and defining λ as follows: If v is a vertex of the fundamental chamber C, then the face of C opposite v has stabilizer $\langle s \rangle$ for some $s \in S$; set $\lambda(v) = s$. This defines λ on the vertices of C, and we use the W-action to extend λ to all the vertices.

For a second situation where there is a canonical labelling, suppose Σ is the barycentric subdivision of a cell complex X, as in Example (c), for instance. Then Σ has one vertex v for every cell F of X, and we set $\lambda(v) = \dim F$. Thus $I = \{0, \ldots, n-1\}$ in this case.

4. Types of galleries

Let Σ be a labelled chamber complex. Then we can use the labelling to refine the adjacency relation on the set $\mathcal{C}$ of chambers: Let I be the set of labels. Then any codimension 1 simplex of Σ has type $I - \{i\}$ for some $i \in I$. Given $i \in I$, two chambers of Σ will be called *i-adjacent* if they have the same face of type $I - \{i\}$. This is an equivalence relation, unlike the ordinary adjacency relation.

Note that two distinct adjacent chambers are i-adjacent for a unique i. Hence a non-stuttering gallery $\Gamma = (C_0, \ldots, C_l)$ has a well-defined *type* $\mathbf{i} = (i_1, \ldots, i_l)$, such that C_{j-1} and C_j are i_j-adjacent for $j = 1, \ldots, l$. We will call $\mathbf{i}$ a *word*, or an *I-word*, even though there is no group under discussion at the moment. In case Σ is thin (i.e., every codimension 1 simplex is a face of exactly two chambers), the assignment $\Gamma \mapsto \mathbf{i}$ yields a bijection from the set of non-stuttering galleries starting at a given chamber C to the set of I-words.

For a familiar example of this, suppose $\Sigma = \Sigma(W, S)$ with its canonical labelling, and let C be the fundamental chamber. One can easily check that the bijection just described coincides with the one mentioned in §1.

5. The Coxeter matrix

Let Σ be a Coxeter complex with an arbitrary labelling $\lambda : V(\Sigma) \to I$. Given $i, j \in I$ with $i \neq j$, choose a simplex σ of type $I - \{i, j\}$. Then the link of σ in Σ is a rank 2 Coxeter complex, hence it is a $2m$-gon for some m ($2 \leq m \leq \infty$). This number m depends only on $\{i, j\}$, and not on the choice of σ, since the group of type-preserving automorphisms of Σ is transitive on the simplices of any given type. Hence we may write

$$m = m_{ij}.$$

If we further set $m_{ii} = 1$, then the resulting matrix $M = (m_{ij})$ is called the *Coxeter matrix* of Σ (with respect to λ). This terminology is explained by the following result:

Proposition. *Let $\Sigma = \Sigma(W, S)$ with its canonical labelling. Then its Coxeter matrix M is the same as the Coxeter matrix of (W, S) in the usual sense, i.e., $m_{s,t}$ is the order of st.*

Proof. To compute the Coxeter matrix, it suffices to consider codimension 2 faces of the fundamental chamber C. Let σ be such a face, and let its type be $S - \{s, t\}$. We must show that the link of σ is a $2m$-gon, where m is the order of st. Now we know that the link of σ is the Coxeter complex associated to the stabilizer W' of σ (cf. §1, Example (e)); and one easily checks from the definition of the canonical labelling that W' is the dihedral subgroup $\langle s, t \rangle$, of order $2m$. So the proposition follows from the fact that the Coxeter complex of a dihedral group of order $2m$ is a $2m$-gon. □

For an arbitrary Σ and λ, the proposition motivates us to introduce the Coxeter group $W = W_M$ associated to M, with generating set $S = \{s_i\}$ in 1-1 correspondence with I, and relations $(s_i s_j)^{m_{ij}} = 1$. It should be reasonably clear to the reader that we will then have $\Sigma \approx \Sigma(W, S)$, as stated in Lecture 2 (§1, Proposition 1). But our real reason for introducing W_M is that it provides the right framework for understanding types of galleries. This is the subject of the next section.

6. The W-valued distance function

Let Σ be a labelled Coxeter complex. The set $\mathcal{C}$ of chambers of Σ can be viewed as a metric space (with distance defined by galleries) or as a set with a family of equivalence relations (i-adjacency). We close this "review" by showing that there is a single structure on $\mathcal{C}$ from which one can obtain both the metric and the adjacency relations.

Proposition. *Let Σ be a labelled Coxeter complex, let M be its Coxeter matrix, and let W_M be the associated Coxeter group.*

(1) *There is a unique function $\delta : \mathcal{C} \times \mathcal{C} \to W_M$ with the following property: Let $C_0, \dots, C_l$ be a non-stuttering gallery and let $\mathbf{i} = (i_1, \dots, i_l)$ be its type. Then $\delta(C_0, C_l)$ is the element $s_{i_1} \cdots s_{i_l}$ of W_M represented by $\mathbf{i}$.*

(2) *Suppose $\Sigma = \Sigma(W, S)$ with its canonical labelling. If we identify both $\mathcal{C}$ and W_M with W in the usual way, then $\delta : W \times W \to W$ is given by*

$$\delta(w, w') = w^{-1}w'.$$

Proof. Uniqueness in (1) is obvious. To prove existence, we may assume $\Sigma = \Sigma(W, S)$ with its canonical labelling. We then take the formula in (2) as a definition. A non-stuttering gallery of type $\mathbf{s} = (s_1, \dots, s_l)$ from w to w' in $\Sigma(W, S)$ has the form

$$w, ws_1, ws_1s_2, \dots, ws_1 \cdots s_l = w',$$

whence $\delta(w, w') = s_1 \cdots s_l$. Thus δ satisfies the condition in (1). □

Note that we recover the gallery metric d from δ by

$$d(C, D) = l(\delta(C, D)),$$

where $l : W_M \to \mathbf{Z}$ is the length function with respect to the generating set $S = \{ s_i : i \in I \}$. And we recover the adjacency relations by

$$C \overset{i}{\sim} C' \iff \delta(C, C') \in \langle s_i \rangle.$$

Appendix B. Buildings as W-Metric Spaces

The purpose of this appendix is to sketch a proof of the theorem stated in §3 of Lecture 2. We will base our sketch on the exercises in §IV.4 of [10], which contain all the necessary ideas. The reader who has worked through those exercises should be able to fill in the missing details.

For ease of reference, we restate the theorem to be proved:

Theorem. *Let (W, S) be a Coxeter system with S finite. Given a set $\mathcal{C}$ and a function $\delta : \mathcal{C} \times \mathcal{C} \to W$, the pair $(\mathcal{C}, \delta)$ arises from a building if and only if it satisfies the following four axioms:*

(1) *$\delta(C, D) = 1$ if and only if $C = D$.*
(2) *$\delta(D, C) = \delta(C, D)^{-1}$.*
(3) *If $\delta(C', C) = s \in S$ and $\delta(C, D) = w$, then $\delta(C', D) = sw$ or w. If, in addition, $l(sw) = l(w) + 1$, then $\delta(C', D) = sw$.*
(4) *If $\delta(C, D) = w$, then for any $s \in S$ there is a C' such that $\delta(C', C) = s$ and $\delta(C', D) = sw$. If $l(sw) = l(w) - 1$, then there is a unique such C'.*

1. Proof of the "only if" part

The notation here is as in Lecture 2: Δ is a labelled building with Weyl group W; S is the distinguished generating set of W and is also identified with the set of labels of Δ; $\mathcal{C}$ is the set of chambers of Δ; and $\delta : \mathcal{C} \times \mathcal{C} \to W$ is the "distance function" obtained by considering types of minimal galleries. We need to prove that δ satisfies axioms (1)–(4) above.

(1) and (2) are immediate. To prove (3) and (4), let $\delta(C, D) = w$, let $\mathbf{s} = (s_1, \ldots, s_l)$ be a reduced decomposition of w, and choose a minimal gallery $C = C_0, \ldots, C_l = D$ of type $\mathbf{s}$; this is possible by Exercise 3(c) in the set of exercises cited above. Suppose first that $l(sw) = l(w)+1$. Then for any C' with $\delta(C', C) = s$, the composite gallery $C', C_0, \ldots, C_l$ has reduced type $\mathbf{s}' = (s, s_1, \ldots, s_l)$ so it is minimal (*loc. cit.*, Exercise 1) and $\delta(C', D) = sw$. Both (3) and (4) follow in this case. Now suppose $l(sw) = s(w) - 1$. Then we can choose $\mathbf{s}$ so that $s_1 = s$. We then have $\delta(C, C_1) = s$ and $\delta(C_1, D) = sw$. And if C' is any chamber distinct from C_1 and s-adjacent to C, then the gallery $C', C_1, \ldots, C_l$ is minimal [being non-stuttering and of reduced type $\mathbf{s}$], hence $\delta(C', D) = w$. □

The reader who has done the exercises cited above will note that we have just repeated an argument used in the solution of Exercise 4(b), as outlined in the hint to that exercise. It turns out that the rest of Exercise 4, which was concerned with the construction of apartments in the building Δ, can be done as a formal consequence of the properties of δ. In other words, it depends only on (1)–(4), and not on the fact that $(\mathcal{C}, \delta)$ comes from a building. We will indicate how this is done in the next section. We will then be able to prove the "if" part of the theorem in the following section.

2. The construction of apartments

In this section we assume only that (W, S) is a Coxeter system and that $\mathcal{C}$ is a set with a function $\delta : \mathcal{C} \times \mathcal{C} \to W$ satisfying (1)–(4). If we set $d(C, D) = l(\delta(C, D))$, then it is easy to see that d is a $\mathbf{Z}$-valued metric in the usual sense.

Note also that we have for each $s \in S$ an equivalence relation of *s-adjacency* on $\mathcal{C}$, defined by

$$C \overset{s}{\sim} C' \iff \delta(C, C') \in \langle s \rangle.$$

Note next that, in view of (2), there are analogues of (3) and (4) obtained by interchanging the roles of C and D. The analogue of the first part of (3), for example, is:

(3′) If $\delta(C, D) = w$ and $\delta(D, D') = s \in S$, then $\delta(C, D') = ws$ or w.

This has the following interpretation: Fix $C \in \mathcal{C}$ and let $\rho : \mathcal{C} \to W$ be the function $\delta(C, -)$; then ρ preserves s-adjacency for all $s \in S$. [In case $(\mathcal{C}, \delta)$ comes from a building, one can explain this fact conceptually by choosing an apartment Σ containing C and relating ρ to the retraction $\rho_{\Sigma,C} : \Delta \to \Sigma$. Details are left to the interested reader.]

We turn now to the main goal of this section, which is the construction of strong isometries from W into $\mathcal{C}$. [Recall that a strong isometry is a function that preserves δ, where δ is defined on W by $\delta(w, w') = w^{-1}w'$.] Even though we do not yet know that $(\mathcal{C}, \delta)$ comes from a building, one should think of the image of a strong isometry $W \to \mathcal{C}$ as an apartment. So the following theorem, in effect, proves the existence of apartments containing given subsets of $\mathcal{C}$.

Theorem. *For any subset $\mathcal{D} \subset W$, any strong isometry $\alpha : \mathcal{D} \to \mathcal{C}$ extends to a strong isometry $W \to \mathcal{C}$.*

We confine ourselves to an outline of the proof. The missing details are not difficult and can be found in the hints to Exercise 4 (*loc. cit.*), or the paper of Tits [25], or Ronan's book [18].

The crucial step is to show that if w_1 and w_2 are adjacent, with $w_1 \in \mathcal{D}$ and $w_2 \notin \mathcal{D}$, then α extends to $\mathcal{D} \cup \{w_2\}$. We may assume that $w_1 = 1$ (the "fundamental chamber"), in which case $w_2 = s$ for some $s \in S$. Let $\alpha(1) = C$. We wish to extend α by setting $\alpha(s) = C'$ for a suitable $C' \in \mathcal{C}$ with $\delta(C, C') = s$. The extension will be a strong isometry provided

$$\delta(C', \alpha(w)) = sw$$

for all $w \in \mathcal{D}$. Equivalently, if we define $f : \mathcal{D} \to W$ by $f(w) = s\delta(C', \alpha(w))$, then we want $f(w) = w$ for all w.

[Note, for future reference, that f is distance-decreasing, i.e.,

$$d(f(w), f(w')) \le d(w, w').$$

One sees this by writing $f(w) = s\rho(\alpha(w))$, where $\rho = \delta(C', -) : \mathcal{C} \to W$, and using the fact that ρ preserves the adjacency relations. See the discussion following (3′) above.]

Now we know, since α is a strong isometry, that $\delta(C, \alpha(w)) = w$ for all $w \in \mathcal{D}$. So the second part of axiom (3) implies that, for any choice of C', we will have $f(w) = w$ for all $w \in \mathcal{D}$ such that $l(sw) = l(w) + 1$. Moreover, axiom (4) says that for any $w \in \mathcal{D}$ with $l(sw) = l(w) - 1$, we can find a C' such that the resulting f will satisfy $f(w) = w$ for that particular w. What we must show, then, is that the (unique) C' which works for one such w works for all of them.

Let $\mathcal{E} = \{ w \in \mathcal{D} : l(sw) = l(w) - 1 \}$, and choose C' as above so that f satisfies $f(w) = w$ for at least one $w \in \mathcal{E}$ (unless $\mathcal{E} = \varnothing$, in which case we are already done). Then f is distance-decreasing and satisfies $f(w) = w$ or sw for any $w \in \mathcal{E}$. An easy geometric argument (cf. *loc. cit.*, last part of the hint to Exercise 4(b)) now shows that in fact $f(w) = w$ for all $w \in \mathcal{E}$. □

3. Proof of the "if" part of the theorem

Using the theorem just proved, we can complete the sketch of the proof of our main theorem. Assume, as in §2, that $(\mathcal{C}, \delta)$ satisfies (1)–(4); our task is to construct a building with $(\mathcal{C}, \delta)$ as the associated "W-metric space". We already know what the chambers of the desired Δ should be, and we already know what the adjacency relations should be. To construct the rest of the simplices, fix a subset $S' \subseteq S$, let $W' = \langle S - S' \rangle$, and consider the following relation on $\mathcal{C}$:

$$C \sim D \iff \delta(C, D) \in W'.$$

This is an equivalence relation, and the equivalence classes will be called *simplices of type* S'. [Motivation: We want to specify a simplex σ by giving the set of chambers of which σ is a face.] If $S'' \subseteq S'$, then any simplex σ of type S', viewed as a subset of $\mathcal{C}$, is contained in a unique simplex τ of type S''; we say that τ is the *face* of σ of type S''. It is now straightforward to verify that the set of simplices, with the face relation just defined, is indeed the set of simplices of a simplicial complex Δ.

For any strong isometry $\alpha : W \to \mathcal{C}$, the image of α generates a subcomplex Σ of Δ, called an apartment. One now verifies the building axioms **(B0)** and **(B2)** without much difficulty. [For **(B2)**, use the maps ρ introduced in §2.] And **(B1)** follows from the theorem of §2, since any two-element subset of $\mathcal{C}$ is strongly isometric to a subset of W. □

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