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The Geometry of Finitely Presented Infinite Simple Groups

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Abstract. Let $\mathcal{G}$ be the family of finitely presented infinite simple groups introduced by Higman, generalizing R.J. Thompson's group of dyadic homeomorphisms of the Cantor set. For each $G \in \mathcal{G}$ and each integer $n \geq 1$, an $(n-1)$-connected n-dimensional simplicial complex is constructed, on which G acts with finite stabilizers and with an n-simplex as fundamental domain. This yields homological and combinatorial information about G. As a by-product, one obtains a solution to a problem of Neumann and Neumann.

Introduction

In 1965 R.J. Thompson gave the first example of a finitely presented infinite simple group (cf. [7]). It can be described as a group of homeomorphisms of the Cantor set or, alternatively, as a certain algebraic automorphism group. Higman [6] later introduced an infinite family $\mathcal{G}$ of finitely presented infinite simple groups generalizing Thompson's example. It is shown in [4] that each $G \in \mathcal{G}$ has the homological finiteness property FP_∞, i.e., that $\mathbf{Z}$ admits a resolution by finitely generated free $\mathbf{Z}G$-modules. The proof is topological; it involves the construction of highly connected simplicial complexes X such that G acts on X with finite stabilizers and compact quotient. But the complexes X are complicated, and it is difficult to obtain any information about G from them other than the FP_∞ property. The purpose of the present note is to give a better construction. Recall that a space X is said to be $(n-1)$-*connected* if $\pi_i(X) = 0$ for $i < n$.

MAIN THEOREM. *For any $G \in \mathcal{G}$ and any $n \geq 1$ there is an $(n-1)$-connected n-dimensional simplicial complex X such that G acts on X with finite stabilizers and with an n-simplex as fundamental domain.*

The term "fundamental domain" here is to be understood in the strong sense: There is a closed n-simplex Δ which maps homeomorphically onto the quotient space X/G. Equivalently, every simplex of X is in the G-orbit of a unique face of Δ.

This theorem yields much more homological information about G than I was able to obtain in [4]. An immediate consequence, for instance, is that G is $\mathbf{Q}$-acyclic. The theorem also yields interesting combinatorial information about G: Take $n = 2$; then X is simply-connected, and it follows that G is the direct limit of the diagram

$$\begin{tikzcd} & & C & & \\ & W \arrow[ur] \arrow[dl] & & V \arrow[ul] \arrow[dr] & \\ A & & U \arrow[ll] \arrow[rr] & & B \end{tikzcd}$$

formed by the stabilizers of the vertices and edges of Δ, where all maps are inclusions. Equivalently, G is the free product of the vertex stabilizers A, B, C, amalgamated along their intersections.

This is interesting for two reasons. First, it settles an old question of Neumann and Neumann [8] about embeddings of finite amalgams. See §6 below for details. Secondly, it gives examples to illustrate some geometric notions recently introduced by Gersten and Stallings for "triangles of groups". In particular, the triangles that arise in the present paper are "positively curved", and they behave quite differently from the triangles of "non-positive curvature" studied by Gersten and Stallings.

For simplicity, I will prove the theorem and corollaries stated above only for Thompson's original group G rather than for an arbitrary $G \in \mathcal{G}$. The interested reader can easily generalize the proofs; this requires more complicated notation, but no new ideas.

The paper is organized as follows. §1 contains a brief treatment of triangles of groups, in order to provide the background for our later combinatorial study of Thompson's group G. The definition of G, as an algebraic automorphism group, is then reviewed in §2. This is followed by an optional §3, in which the same group is described as a group of homeomorphisms of the Cantor set. Readers are free to adopt, for the remainder of the paper, whichever definition they prefer. §4 is devoted to an unpublished result of Melanie Stein, which yields a contractible complex X on which G acts with finite stabilizers. It is a subcomplex of a contractible complex that was used in [4]. The main theorem is then proved in §5; the complexes called X in the statement of that theorem are obtained by suitably "truncating" the X of §4. Finally, §§6 and 7 contain corollaries: Combinatorial results are given in §6 and homological results in §7.

I am grateful to G. Higman for telling me about the problem of Neumann and Neumann and pointing out that my results solved it.

1. Triangles of Groups

Recall that the amalgamated free product $G = A*_U B$ is defined whenever one is given a diagram

$$A \leftarrow U \rightarrow B$$

of groups and monomorphisms. By definition, G is the direct limit of this diagram. It is well-known that the canonical maps $A \to G$ and $B \to G$ are injective. Moreover, if we identify A, B, and U with their images in G, then $U = A \cap B$.

In 1948 Hanna Neumann [9] considered a more general situation, where one is given an arbitrary family of groups $A, B, C, \ldots$ and, between any two of them, a common subgroup to amalgamate. We will be interested in the case of three free factors, in which case what we are given is a diagram

$$\begin{tikzcd} & & C & & \\ & W \arrow[ur] \arrow[dl] & & V \arrow[ul] \arrow[dr] & \\ A & & U \arrow[ll] \arrow[rr] & & B \end{tikzcd} \tag{$*$}$$

of groups and monomorphisms. We call such a diagram a *triangle of groups*, and we call A, B, C (resp. U, V, W) the *vertex groups* (resp. *edge groups*) of the triangle. Let G be the direct limit of the diagram $(*)$. Unlike the situation with two free factors, the canonical maps from the vertex groups to G need not be injective. But if they happen to be injective, and if in addition the image of each edge group in G is the intersection of the images of the corresponding vertex groups, then we call the triangle *realizable.* In this case G is said to be the *free product of A, B, and C, amalgamated along U, V, and W*, and we identify all the groups in $(*)$ with their images in G. Following Neumann [9], one sometimes uses the term "generalized amalgamated free product", to avoid confusion with the classical case where there are only two free factors.

REMARK. Readers familiar with Bass–Serre theory [11] will note that $(*)$ can be viewed as a graph of groups (where the underlying graph is the boundary of a 2-simplex). One is therefore tempted to form the fundamental group $\tilde{G}$ of this graph of groups. It is well-defined after a choice of basepoint or maximal tree. One such choice yields the following description of $\tilde{G}$: First construct $H = A *_U B *_V C$; then $\tilde{G}$ is the HNN extension $\langle\, H, t\,;\, t^{-1}W't = W''\,\rangle$, where W' (resp. W'') is the image of W in A (resp. C). Thus the direct limit G that we are studying is the quotient of $\tilde{G}$ obtained by introducing the relation $t = 1$.

Call the triangle $(*)$ *fillable* if it can be completed to a commutative diagram

$$\begin{tikzcd}
 & & C & & \\
 & W \arrow[ur] \arrow[dl] & K \arrow[l] \arrow[r] \arrow[d] & V \arrow[ul] \arrow[dr] & \\
A & & U \arrow[ll] \arrow[rr] & & B
\end{tikzcd}$$

in which all maps are monomorphisms and each of the three squares has the following property: The image of K in the vertex group is the intersection of the images of the two edge groups. As Neumann [9] pointed out, fillability is a necessary condition for realizability; for if $(*)$ is realizable, then we can fill the triangle with $K = A \cap B \cap C$. But there are examples in [9] which show that fillability is not sufficient for realizability.

We turn now to connections with topology. Recall that Serre [11] has given a topological interpretation of ordinary amalgamated free products, in terms of group actions on trees with a 1-simplex as fundamental domain. There is an analogous result, due to Soulé [12] and Behr [1], for generalized amalgamated free products. In the case at hand, where there are three factors, the result is that amalgamated free product decompositions correspond to group actions on 1-connected 2-dimensional simplicial complexes with a 2-simplex as fundamental domain. More precisely, suppose G is the amalgamated free product associated to a realizable triangle $(*)$ as above, and let X be the (essentially unique) simplicial 2-complex such that G acts on X with a 2-simplex Δ as fundamental domain and with A, B, and C as the stabilizers of the vertices of Δ. Thus the vertex set of X is $G/A \amalg G/B \amalg G/C$, and a collection of vertices is a simplex if and only if the corresponding cosets have a non-empty intersection in G. Then X is 1-connected ([1], Satz 1.2). Conversely, if a group G acts on a 1-connected simplicial 2-complex with a 2-simplex Δ as fundamental domain, then G is the free product of the stabilizers of the vertices of Δ, amalgamated along their intersections (cf. [12] or [3]).

We conclude this section by stating some results of Gersten and Stallings [unpublished] concerning triangles of groups with "non-positive curvature". Gersten and Stallings begin by assigning an angle to each of the corners of the triangle $(*)$. To define the angle at A, for instance, let Y be the (essentially unique) 1-dimensional simplicial complex such that A acts on Y with an edge e as fundamental domain and with U and W as the stabilizers of the vertices of e. Let k be the smallest integer ≥ 3 such that Y contains a k-gon, if such an integer exists; otherwise, let $k = \infty$. Then the angle at A is defined to be $2\pi/k$. [The motivation for this comes from the fact that if the triangle is realizable and X is the associated 2-complex, then Y is the link in X of the vertex of Δ corresponding to A.]

The results of Gersten and Stallings concern triangles which are fillable and have angle sum $\leq \pi$. Their first theorem is that such a triangle is always realizable. They go on to prove a number of results about the generalized amalgamated free product G and the associated 2-complex X. For example, they show that X is contractible. In fact, they endow X with a geometric structure, making it, in some sense, a space of non-positive curvature in which any two points can be joined by a unique geodesic. They then use this geometric structure to prove that every bounded subgroup of G is conjugate to a subgroup of one of the vertex groups A, B, C. "Bounded" here means that there is an integer n such that every element of the given subgroup can be expressed as a product $x_1 \cdots x_l$ with $l \leq n$ and $x_i \in A \cup B \cup C$.

Suppose we add the hypothesis that the vertex groups are finite. Then the previous paragraph yields, among other things, the following two results: (a) G has only finitely many conjugacy classes of finite subgroups. (b) Any torsion-free subgroup of G has cohomological dimension at most 2.

We proceed now to Thompson's group. As we will see, it is the generalized amalgamated free product associated to a realizable triangle of finite groups. But Thompson's group has finite subgroups of arbitrarily large order, and it has torsion-free subgroups of infinite cohomological dimension. Our triangle, then, will necessarily have angle sum greater than π.

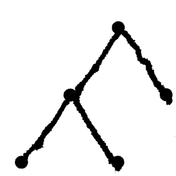
2. Thompson's Group

We begin with a very brief review of the definition of Thompson's group G as an algebraic automorphism group. See [6] and §4 of [4] for more details, further references, and historical remarks. See also §3 below for an alternative treatment, where G is viewed as a group of homeomorphisms of the Cantor set.

Consider the algebraic system consisting of a set S together with a bijection $\alpha : S \to S \times S$. For lack of a better name, we will simply call S an *algebra* in what follows. Let α_0 and α_1 be the components of α. They are unary operators $S \to S$, which we write on the right. Thus α is given by $x \mapsto (x\alpha_0, x\alpha_1)$. As an aid to the intuition, we will sometimes call $x\alpha_0$ and $x\alpha_1$ the two *halves* of x.

Let S be the free algebra on a single generator x. Then S has bases of arbitrary finite cardinality. For example, it has a 2-element basis $\{x\alpha_0, x\alpha_1\}$ and a 3-element basis $\{x\alpha_0\alpha_0, x\alpha_0\alpha_1, x\alpha_1\}$. These examples illustrate a general method for constructing bases: Given an r-element basis L and an element $y \in L$, we get an $(r+1)$-element basis M by replacing y by its two halves y_0, y_1. One says that M is a *simple expansion* of L. Conversely, if we start with a basis M and an ordered pair (y_0, y_1) of distinct elements of M, then we obtain a basis L by replacing y_0 and y_1 by $y = \alpha^{-1}(y_0, y_1)$. The basis L is called a *simple contraction* of M. Given two bases L, M, then, M is a simple expansion of L if and only if L is a simple contraction of M.

We now iterate these constructions: We say that M is an *expansion* of L and that L is a *contraction* of M if there is a sequence $L = L_0, \ldots, L_d = M$ $(d \geq 0)$ with L_i a simple expansion of L_{i-1} for $1 \leq i \leq d$. It is shown in [6] that any two bases have a common expansion. In other words, if L and M are arbitrary bases, then one can get from L to M by doing an expansion followed by a contraction. In particular, any basis can be constructed as a contraction of an expansion of the original basis $\{x\}$. Note, in this connection, that expansions of $\{x\}$ are very concrete objects; namely, they correspond to finite, rooted, binary trees. For example, the 3-element basis given in the previous paragraph is represented by the tree



Thompson's group G is now defined to be the group of automorphisms of S. This is the group called $G_{2,1}$ in [6] and [4]. To construct examples of elements of G, choose two expansions L, M of $\{x\}$ with the same cardinality, and choose an arbitrary bijection $L \to M$; then this bijection extends uniquely to an automorphism g of S. Moreover, every $g \in G$ can be described in this way.

By taking $L = M$ above, we see that G contains, for every positive integer n, the symmetric group on n letters. Consequently, every finite group can be embedded in G. In fact, every countable locally finite group can be embedded in G ([6], Theorem 6.6).

On the other hand, G also contains interesting torsion-free subgroups. For example, it contains the group F studied in [5] (and called $F_{2,1}$ in [4]). In particular, G contains torsion-free subgroups of infinite cohomological dimension.

3. Dyadic Homeomorphisms of the Cantor Set

This section, which may be omitted, presents an alternative view of Thompson's group.

Let C be the Cantor set of infinite sequences $a = (a_j)_{j \geq 1}$ with $a_j \in \{0, 1\}$. It is topologized as the product of infinitely many discrete two-point spaces. Note that there is a decomposition $C = C_0 \amalg C_1$, where C_i is the set of sequences a with $a_1 = i$. Each C_i is canonically homeomorphic to C, so we can iterate the process, thereby expressing C as the union of n copies of itself for any positive integer n. For example, we can partition C_0 into two Cantor sets C_{00} and C_{01} to get

$$C = C_{00} \amalg C_{01} \amalg C_1.$$

Finite partitions of C obtained in this way will be called *standard partitions.* There is one such partition for every finite, rooted, binary tree. Thus standard partitions correspond to expansions of $\{x\}$ in the notation of §2. The number of subspaces occurring in a partition will be called its *rank.* This is the same as the cardinality of the associated basis of S. It is also the same as the number of leaves of the associated tree, where a leaf is a node with no descendants.

Suppose now that we are given two standard partitions $L = (L_i)_{i \in I}$ and $M = (M_j)_{j \in J}$ with the same rank. For example, we could take $L = \{C_{00}, C_{01}, C_1\}$ and $M = \{C_0, C_{10}, C_{11}\}$. Choose a bijection between I and J. Then we can construct a homeomorphism $g : C \to C$ which maps each L_i to the corresponding M_j by the canonical homeomorphism. This makes sense because each L_i is canonically homeomorphic to C, as is each M_j. Any g constructed in this way will be called a *dyadic homeomorphism* of C.

The dyadic homeomorphisms form a group under composition, and this group G is Thompson's finitely presented infinite simple group, as described for instance in [7]. It is not hard to see that G is isomorphic to the group called G in §2, but we will not make any use of this isomorphism.

We close this section by discussing partitions of C more general than the standard ones. These play the role of bases of S more general than those which are expansions of $\{x\}$. We will need a precise definition of the word "partition", which we have been using informally up to now: A *partition* of C of rank n is a pair (L, H), where L is a collection of n pairwise disjoint subspaces of C whose union is C, and H is a collection of homeomorphisms $h : C \to D$, one for each $D \in L$. Roughly speaking, then, a partition exhibits C as the disjoint union of finitely many copies of itself. We will often suppress H from the notation and simply say that L is a partition.

Note that the standard partitions discussed earlier yield partitions in the present sense, since each subspace occurring in a standard partition is canonically homeomorphic to C. We wish to extend to arbitrary partitions the subdivision process used earlier to construct standard partitions.

Given a partition L and an element $D \in L$, we can write $D = hC_0 \amalg hC_1$, where $h : C \to D$ is the homeomorphism associated to D. Then each hC_i is canonically homeomorphic to C, so we obtain a new partition M by replacing D by its two "halves" hC_0 and hC_1. We say that M is a *simple expansion* of L. Conversely, given a partition M and an ordered pair (D_0, D_1) of distinct elements of M, we can construct a *simple contraction* L of M by replacing D_0 and D_1 by $D = D_0 \amalg D_1$, which is canonically homeomorphic to C. As in §2, we can iterate these constructions to obtain more general notions of expansion and contraction.

In contrast to the situation of §2, two partitions need not have a common expansion. We remedy this by restricting attention to a subset of the set of all partitions: A partition is called *admissible* if it is a contraction of some standard partition. It is easy to check that any two admissible partitions have a common expansion. The interested reader can now construct a bijection between the set of admissible partitions of C and the set of bases of S.

Finally, we wish to define an action of G on the set of admissible partitions. Note first that the full homeomorphism group of C acts on the set of all partitions: Given a partition (L, H) and a homeomorphism $g : C \to C$, there is a new partition (gL, gH), where $gL = \{gD : D \in L\}$ and

$gH = \{ gh : h \in H \}$. Here gh is to be interpreted as the composite

$$C \xrightarrow{h} D \xrightarrow{g|D} gD.$$

This action has the property that the stabilizer of a rank n partition is isomorphic to the symmetric group on n letters.

One can now verify:

PROPOSITION. *A partition is admissible if and only if it is in the G-orbit of a standard partition. In particular, the set of admissible partitions is invariant under the action of G.* □

4. A contractible space for G

Our starting point is the set $\mathcal{B}$ of bases of S. [Readers of §3 may, if they wish, work instead with the set of admissible partitions of the Cantor set.] As in [4], we view $\mathcal{B}$ as a poset, where $L \leq M$ if M is an expansion of L. This poset structure is compatible with the obvious action of G on $\mathcal{B}$, i.e., G acts by poset automorphisms. Let $|\mathcal{B}|$ be the simplicial complex associated to $\mathcal{B}$; its simplices are the finite chains $L_0 < \cdots < L_n$ in $\mathcal{B}$. This is the G-complex with finite stabilizers which was used in [4] to study G. Since $\mathcal{B}$ is a directed set, $|\mathcal{B}|$ is contractible. The purpose of the present section is to present a result of Melanie Stein [unpublished], which provides a contractible G-invariant subcomplex of $|\mathcal{B}|$.

Given a basis L of S, an *elementary expansion* of L is a basis M obtained by choosing a subset $L' \subseteq L$ and replacing each $y \in L'$ by its two halves y_0, y_1. Thus we can get from L to M by a sequence of simple expansions, where at each stage we expand some $y \in L$ rather than one of the halves introduced by an earlier expansion. We write $L \preceq M$ if M is an elementary expansion of L, and we write $L \prec M$ if, in addition, $L \neq M$. [Warning: The relations "$\preceq$" and "$\prec$" are not transitive.]

Call a simplex $L_0 < \cdots < L_n$ *elementary* if $L_0 \preceq L_n$. This implies that $L_i \preceq L_j$ for $i \leq j$. Hence any face of an elementary simplex is elementary, and the elementary simplices form a (G-invariant) subcomplex $X \subset |\mathcal{B}|$.

THEOREM 1 (M. STEIN). *The complex X of elementary simplices is contractible.*

We will use, in the proof, the standard notation for intervals in a poset. For example, the open interval (L, M) is defined by

$$(L, M) = \{ N \in \mathcal{B} : L < N < M \}.$$

The closed interval $[L, M]$ and the half-open intervals $[L, M)$ and $(L, M]$ are defined similarly. The following lemma is the key step in the proof.

LEMMA. *If M is a non-elementary expansion of L, then $\left|(L, M)\right|$ is contractible.*

PROOF: For any expansion N of L, let N_0 be the largest element of $[L, N]$ such that $L \preceq N_0$; it is obtained by taking L', in the definition of "elementary expansion", to consist of all elements $y \in L$ which get expanded in the passage from L to N. Note that we have $N_0 \in (L, M)$ for any $N \in (L, M]$. The inequalities $N \geq N_0 \leq M_0$ now yield a "conical" contraction of (L, M), cf. [10], 1.5. □

Note that the complexes $\left|[L, M]\right|$, $\left|[L, M)\right|$, and $\left|(L, M]\right|$ are also contractible, for trivial reasons, since the intervals in question all have a largest or smallest element.

PROOF OF THEOREM 1: Since $|\mathcal{B}|$ is already known to be contractible, it suffices to show that we can obtain $|\mathcal{B}|$ from X by a sequence of adjunctions which do not change the homotopy type. To this end, we construct $|\mathcal{B}|$ by successively adjoining the subcomplexes $\left|[L, M]\right|$ with M a non-elementary expansion of L. We do these adjunctions by induction on $r(M) - r(L)$, where $r(\,)$ denotes the cardinality of a basis. Thus the part of $\left|[L, M]\right|$ already present at the time of the adjunction is $\left|[L, M) \cup (L, M]\right|$, which is the suspension of $\left|(L, M)\right|$ and hence is contractible by the lemma. Since $\left|[L, M]\right|$ is also contractible, we conclude that the adjunction has no effect on the homotopy type. □

REMARK. Everything in this section goes through without change if we fix an integer r and replace $\mathcal{B}$ by the set of bases of cardinality $\geq r$.

5. Proof of the Main Theorem

Recall from the introduction that our goal is to construct for any $n \geq 1$ an $(n-1)$-connected n-dimensional simplicial complex on which G acts with finite stabilizers and with an n-simplex as fundamental domain. We will do this by "truncating" the complex X of elementary simplices. Given integers p, q with $1 \leq p \leq q$, let $X_{p,q}$ be the full subcomplex of X generated by the vertices L with $p \leq r(L) \leq q$. Here, as above, $r(\,)$ denotes the cardinality of a basis. Note that $X_{p,q}$ is G-invariant. Note also that the dimension of $X_{p,q}$ satisfies $\dim X_{p,q} \leq q - p$, with equality if $q \leq 2p$.

Now fix an integer $n \geq 1$, and consider the G-complex $X_{p,p+n}$ for $p \geq n$. It is n-dimensional. Associated to any simplex $L_0 < \cdots < L_m$ of this complex is a sequence of integers $r(L_0) < \cdots < r(L_m)$, called the *type* of the simplex. It is easy to check that two simplices are G-equivalent if and only if they have the same type. Since every n-simplex has exactly one face of each possible type, we conclude that the action of G on $X_{p,p+n}$ admits an n-simplex as fundamental domain. The main theorem now follows from:

THEOREM 2. *There is an integer p_0 (depending on n) such that $X_{p,p+n}$ is $(n-1)$-connected for $p \geq p_0$.*

The proof will make use of a family of simplicial complexes K_r $(r \geq 2)$, defined as follows: The vertices of K_r are the ordered pairs (a, b) with $a, b \in \{1, \ldots, r\}$ and $a \neq b$; a collection $\{(a_0, b_0), \ldots, (a_m, b_m)\}$ of such vertices is a simplex if $\{a_i, b_i\} \cap \{a_j, b_j\} = \emptyset$ for $i \neq j$. The complexes K_r appeared in [4], where it was shown that the connectivity of K_r tends to ∞ with r (cf. [4], Lemma 4.20). In other words:

LEMMA. *There is an integer r_0 (depending on n) such that K_r is $(n-1)$-connected for $r \geq r_0$.* $\square$

PROOF OF THEOREM 2: Fix p and consider the sequence of inclusions

$$X_{p,p+n} \subset X_{p,p+n+1} \subset X_{p,p+n+2} \subset \cdots .$$

The union is the full subcomplex of X generated by the vertices L with $r(L) \geq p$. This union is contractible by the remark at the end of §4. So $X_{p,p+n}$ will be $(n-1)$-connected if the inclusion $X_{p,q} \hookrightarrow X_{p,q+1}$ induces an isomorphism $\pi_i(X_{p,q}) \xrightarrow{\approx} \pi(X_{p,q+1})$ for all $i < n$ and all $q \geq p+n$. Now this inclusion is obtained by adjoining, for each basis M with $r(M) = q+1$, a cone over the link of M in $X_{p,q}$. Fix M and let Y be this link. Its vertices are the bases L with $L \prec M$ and $r(L) \geq p$, and its simplices are the chains $L_0 < \cdots < L_m$ of such bases.

We can describe a basis $L \prec M$ by specifying the pairs of elements of M which are contracted to get L. Thus Y has one vertex for every non-empty set $P \subset M \times M$ satisfying:

(1) If $(a, b) \in P$, then $a \neq b$.
(2) $\{a, b\} \cap \{c, d\} = \emptyset$ for any two distinct pairs $(a, b), (c, d) \in P$.
(3) P has cardinality $\leq q + 1 - p$.

And the simplices of Y correspond to chains $P_0 \subset \cdots \subset P_m$ of such sets P. Now sets P satisfying (1) and (2) are the same as simplices of the complex K_{q+1} defined above. And condition (3) says that the simplex has dimension at most $q-p$. So our complex Y, which consists of chains of such simplices, is the barycentric subdivision of the $(q-p)$-skeleton of K_{q+1}. The lemma now implies that Y is $(n-1)$-connected if $q+1 \geq r_0$ and $q-p \geq n$.

This last inequality is vacuous, since we are only considering integers $q \geq p+n$. And the first will be satisfied for all $q \geq p+n$ provided $p+n+1 \geq r_0$. So if we take $p_0 = r_0 - n - 1$ and $p \geq p_0$, then the complexes Y that arise above will all be $(n-1)$-connected. Attaching a cone over such a Y does not affect π_i for $i < n$; hence $X_{p,p+n}$ is $(n-1)$-connected, as required. □

REMARK. K. Vogtmann [private communication] has shown that one can take $r_0 = 3n+2$, which gives $p_0 = 2n+1$. Thus $X_{3,4}$ is connected, $X_{5,7}$ is 1-connected, $X_{7,10}$ is 2-connected, etc. Here is a proof of Vogtmann's result for the case $n = 2$:

We must show that K_r is 1-connected for $r \geq 8$. It is trivial to verify that K_r is connected, so the content of the assertion is that any closed edge path is null-homotopic. To prove this, it suffices to show that any edge path of length 3 is homotopic, relative to its endpoints, to a shorter path. Let v_0, v_1, v_2, v_3 be the vertices of an edge path of length 3. If v_0 and v_2 are joined by an edge in K_r (i.e., if the corresponding 2-element subsets of $\{1, \ldots, r\}$ are disjoint), then the first three vertices of our path are the vertices of a 2-simplex, and the path can be shortened to v_0, v_2, v_3. A similar remark applies to v_1 and v_3. So we may assume that v_0 and v_2 have an element of $\{1, \ldots, r\}$ in common, and similarly for v_1 and v_3. But then the entire edge path involves at most 6 elements of $\{1, \ldots, r\}$. Since $r \geq 8$, we can now find a vertex v such that the given path is in the star of v, hence it is homotopic to v_0, v, v_3.

6. A Combinatorial Description of G

We specialize now to the case $n = 2$. As we have just proved, the 2-complex $X_{p,p+2}$ is 1-connected for any $p \geq 5$, and G acts on it with a 2-simplex Δ as fundamental domain. Take $p = 5$, for instance. Then the vertices of Δ can be written in the form

$$\begin{aligned} L_5 &= \{a, b, c, d, e\} \\ L_6 &= \{a, b, c, d, e_0, e_1\} \\ L_7 &= \{a, b, c, d_0, d_1, e_0, e_1\}, \end{aligned}$$

where $d_i = d\alpha_i$ and $e_i = e\alpha_i$. The stabilizers of these vertices are isomorphic to Σ_5, Σ_6, and Σ_7, respectively, where Σ_r is the symmetric group on r letters. It is a simple matter to work out the intersections of these stabilizers. The results stated in §1 now yield:

THEOREM 3. *There is a realizable triangle of groups*

$$\begin{array}{ccccc} & & \Sigma_7 & & \\ & \nearrow & & \nwarrow & \\ & \Sigma_3 \times \Sigma_2 & & \Sigma_5 & \\ \swarrow & & & & \searrow \\ \Sigma_5 & \longleftarrow & \Sigma_4 & \longrightarrow & \Sigma_6 \end{array}$$

whose amalgamated free product is Thompson's infinite simple group. □

The maps in this triangle can be described as follows. Regard Σ_r as the group of permutations of $\{1, \ldots, r\}$. The maps $\Sigma_4 \hookrightarrow \Sigma_r$ $(r = 5, 6)$ are then the standard inclusions, obtained by letting Σ_4 permute $\{1, 2, 3, 4\} \subset \{1, \ldots, r\}$. Similarly, the maps $\Sigma_5 \hookrightarrow \Sigma_r$ $(r = 6, 7)$ are obtained by letting Σ_5 permute $\{1, 2, 3, r-1, r\}$. Finally, $\Sigma_3 \times \Sigma_2$ is embedded in Σ_5 in the standard way, with the first factor acting on $\{1, 2, 3\}$ and the second factor acting on $\{4, 5\}$; but the embedding $\Sigma_3 \times \Sigma_2 \hookrightarrow \Sigma_7$ is not of the standard type. The first factor permutes $\{1, 2, 3\}$, but the non-trivial element of the second factor, instead of mapping to a transposition, maps to the product $(46)(57)$ of two transpositions.

REMARK. It is easy to describe links of vertices in $X_{5,7}$, from which one can compute the Gersten–Stallings angles of the triangle above (cf. §1). These angles turn out to be $\pi/3$, $\pi/2$, and $\pi/3$ at the vertices stabilized by Σ_5, Σ_6, and Σ_7, respectively. Note that the sum of these angles is greater than π.

Finally, we explain how Theorem 3 settles a question of Neumann and Neumann [8]. Let A, B, C denote the vertex groups in the triangle above, and let H be their union $A \cup B \cup C$ in G. We view H as a set with a partially defined multiplication, where the product of two elements is defined if at least one of the groups A, B, C contains both of them. In the language of [8], H is an *amalgam* of the groups A, B, C. In general, an amalgam of groups need not be embeddable in a group; but our amalgam is embedded in Thompson's group, by construction.

The question posed in [8], p. 255, is the following: If an amalgam of groups is finite and is embeddable in a group, is it necessarily embeddable in a finite group? The following corollary of Theorem 3 shows that the answer is "no":

COROLLARY. *The finite amalgam H is not embeddable in a finite group.*

PROOF: Since G is the amalgamated free product of A, B, C, it follows that the inclusion $H \hookrightarrow G$ is the universal homomorphism from H to a group. So if H could be embedded in a finite group, then it would embed in a finite quotient of G. But G is infinite and simple, so its only finite quotient is the trivial group. $\square$

Note that this is not just an isolated example. For we could replace 5 by any $p \geq 5$ above, thereby getting infinitely many triangles with similar properties. Moreover, as stated in the introduction, we could replace G by any member of Higman's family of finitely presented infinite simple groups.

7. Homology Calculations

THEOREM 4. *Thompson's group G is $\mathbf{Q}$-acyclic, i.e., $H_i(G, \mathbf{Q}) = 0$ for all $i > 0$.*

PROOF: Fix n, and let $Y = X_{p,p+n}$ for large p. Consider the equivariant homology $H_*^G(Y, \mathbf{Q})$, as defined for instance in [2], §VII.7. Since the stabilizer of every simplex of Y is finite, this homology is isomorphic to $H_*(Y/G)$ (cf. [2], §VII.7, Exercise 2). But Y/G is a simplex, so we have $H_i^G(Y, \mathbf{Q}) = 0$ for $i > 0$. On the other hand, Y is $(n-1)$-connected, so the canonical map $H_i^G(Y) \to H_i(G)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$ (cf. [2], VII.7.2). Hence $H_i(G, \mathbf{Q}) = 0$ for $i \leq n$. Since n is arbitrary, this completes the proof. $\square$

REMARK. There is some evidence to suggest that G might be $\mathbf{Z}$-acyclic. For example, I have proven the following:

(1) $H_i(G) = 0$ for $i = 1, 2, 3$. In particular, the Schur multiplier of G is trivial.

(2) Let Σ_r be embedded in G as the stabilizer of an r-element basis, and let k be any field. Then the induced map $H_i(\Sigma_r, k) \to H_i(G, k)$ is the zero map for all $i > 0$.

(3) Let F be the torsion-free subgroup of G mentioned at the end of §2. Then the inclusion $F \hookrightarrow G$ induces the zero map $H_i(F) \to H_i(G)$ for all $i > 0$.

Assertion (1) is proved by spectral sequence computations. The proofs of (2) and (3), however, are more interesting; they are based on the fact that there is an embedding $G \times G \hookrightarrow G$, which induces a ring structure on $H_*(G)$. Details will be given elsewhere.

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