Playing with Surfaces:
Spheres, Monkey Pants, and Zippergons

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Abstract

We describe a process, inspired by clothing design, of smoothing an octahedron to form a round sphere. This process can be adapted to construct many different surfaces out of paper and craft foam.

Introduction

(a) *Paper and tape model*

(b) *Pattern*

Figure 1: *Octahedral Sphere*

This excursion from the world of topology and geometry into the art of shaping surfaces in the real world began with a letter to Bill Thurston (author number two) from Dai Fujiwara, director of design for the Issey Miyake fashion house of Tokyo. Fujiwara had read an account of Thurston's use of tree leaves, leafy vegetables, orange peels, etc. in teaching geometry to students both elementary and advanced. It happened that Fujiwara had used oranges in a very similar way in a presentation to students in Tokyo who asked him to explain how to design clothing. Fujiwara arranged to visit Cornell, and after much discussion between Fujiwara and Thurston, the Issey Miyake 2010 Spring season was themed on the eight geometries for 3-dimensional topology.

The Paris show precipitated a series of encounters and projects afterwards, including a joint workshop between the departments of Mathematics and Fiber Science & Apparel Design at Cornell. For that workshop as well as in Thurston's seminar, we worked on devising schemes for designing pattern pieces to fit arbitrary shapes, including but not limited to human forms. It was a very interesting but humbling experience, because our initial assumption that familiar theoretical principles of differential geometry would do most of the work was misleading. Clothing is usually constructed from fabric which has a lot of stretchiness or give, at least in diagonal directions (on the bias). We have focused instead on using more dimensionally stable materials. Most clothing is constructed using one of a few time-tested patterns of traditional seams. We wanted novelty.

Figure 2: *Marking a surface*

The initial idea was this: For any curve on any smooth surface, there are curves in the Euclidean plane so that a neighborhood of the curve on the surface can be matched to a neighborhood of the curve in the plane in a way that their metrics (measurements of distances along the surface) agree up to first order. This principle extends to any tree-like drawing on a smooth surface; sometimes when the curve or tree is mapped to the plane, it might cross itself, but that problem can be solved either by patching or by modifying the choice of curves. It's not hard to construct a tree (or small number of trees), that comes reasonably close to every point on the surface. We devised a method to mark a tree on the desired surface, represented by say a dressmaker's dummy, then lay it out appropriately on a flat sheet. When we tried this, we soon learned the big problem: This method created patterns for which the lengths of the two sides of the seam did not match. In an area of positive curvature (as over a shoulder or breast), the boundary of say a 3-inch neighborhood of the tree is much shorter than in the Euclidean case. In an area of negative curvature, the boundary is longer than in the Euclidean case. Without attention to this mismatch, the pieces just don't work very well.

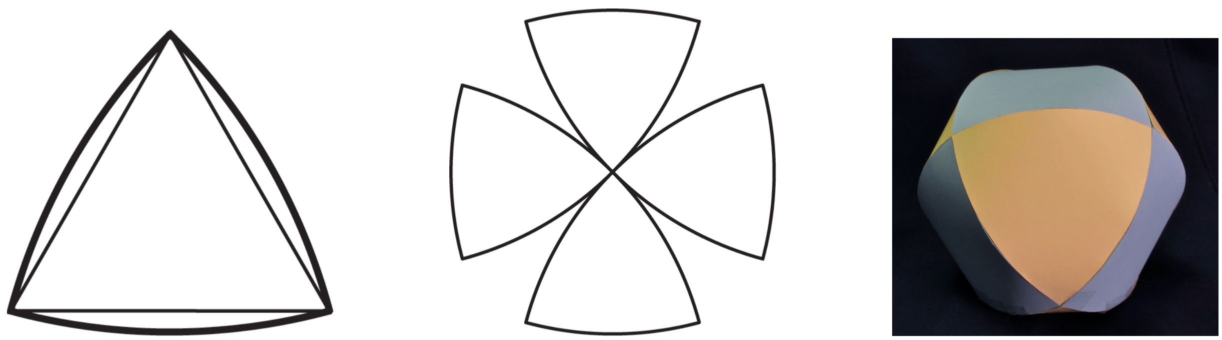
This problem doesn't show up in designing patterns for surfaces of constant curvature, so we decided to spend some effort on the easier problem of designing good patterns for surfaces of constant curvature, starting with the sphere. The results of our first construction can be found in Figure 1.

Spheres

The model pictured in Figure 1 was developed from the octahedron. The Gauss-Bonnet theorem tells us the sphere has total (Gaussian) curvature equal to 4π. This curvature is evenly distributed over the sphere. In the octahedron, as with all polyhedra, the curvature is concentrated at the vertices. Four equilateral triangles, which have angles measuring $60°$, meet at a vertex to give a total angle sum of $240°$. There are $360°$ at a vertex in a polyhedral pattern on a round sphere. The missing angle, $360° - 240° = 120°$, represents curvature concentrated at a vertex. The six vertices of the octahedron give a total angle defect of $6 \times 120° = 720°$, or 4π, as the Gauss-Bonnet theorem predicted.

The first step in the smoothing process is to push the curvature away from the vertices into the edges by increasing the angles of the triangles so that the total angle sum at a vertex is $360°$. Since there are four triangles at a vertex, we need to create $90°$ triangles. This is done by replacing the sides of the triangles with circle arcs which come from a $30°$ sector of a circle. See Figure 3a. In Figure 3b we show four fattened triangles about a vertex to illustrate that they fit snugly; there is no longer any missing angle at the vertex. This pattern works well for spheres made out of cloth, as tested by Margaret Thurston, Bill's mother, who has sewn a number of geometric and topological models based on this principle. However, you can see from the taped up model, Figure 3c, this initial smoothing process does not provide very satisfactory results in the paper model. The constructed sphere is still quite flat.

To improve the design, we need to distribute the curvature over a larger portion of the surface. This can be achieved by making long, meandering seams that have the same amount of total bending as the circle



(a) 90° *triangle* (b) *Four triangles* (c) *Fat octahedron*

Figure 3: *Fat octahderon*

arcs. The basic idea is this. Start with an angle function which maps an interval into the unit circle. Given an initial starting point in the plane, the integral of this function defines a smooth curve, parameterized by arc length, which has a tangent vector equal to the angle function. By modifying compositions of trig functions we can create an angle function whose associated curve has a nice meandering shape, and total curvature equal to 0. (See column 1 of Figure 4). To uniformly bend the curve, add a linear term to the angle function. The total bending of the curve created by integrating the altered angle function is determined by the slope of the linear term. This process is illustrated in Figure 4. The rows show how these curves bend as the total curvature is increased; the greater the curvature the more spreading of the fingers. Note that second two entries in the bottom row are simply arcs of circles.

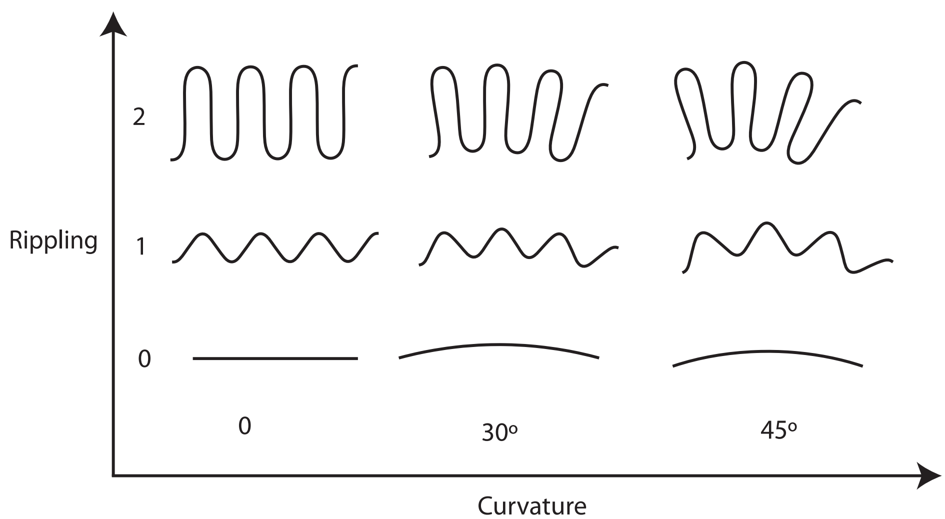


Figure 4: *Bending Curves*

The pattern in Figure 1 was designed from a curve similar to that of the rippling 2, curvature 30°. Three copies of this curve were connected to make the triangular pattern (note the three-fold symmetry). Experimentation revealed that curves with this shape make very round spheres. Surprisingly, the number of ripples is not as important as the distribution of area for achieving smoothness. It is necessary to minimize large blocks of area; every point in the pattern piece should be close to the boundary. For example, the only flatness in the Figure 1 model comes from the small triangular patch in the center of the pattern. Contrast this with an intermediate octahedral sphere, found in Figure 5a. This sphere was created from a pattern made from curves with less rippling.

Figure 5b shows a tetrahedron that has been smoothed. Only three triangles meet at a vertex in a tetrahedron, so for flat vertices one must make even fatter triangles with 120° angles. Since only four pieces are needed, this model assembles quickly, and also produces a very round finished model. Using the bending techniques described one can smooth any of the platonic solids to make sphere models. In fact, the method can be adapted to make surfaces of negative curvature as well, as seen in the Monkey pants model, Figure 5c. This example was constructed from 4 hexagonal pattern pieces, and has tetrahedral symmetry. (The fourth "tail" hole is not visible in the photograph.) The negative curvature can be seen from the fact that the fingers in the pattern come together, rather than spreading apart. Although Gaussian curvature is intrinsic to the surface, curvature of curves in the plane has a sign which depends on a choice of normal vector. If we reflect the curves seen in Figure 4 vertically we see curves with negative bending of the same magnitude.

(a) *Intermediate octahedral sphere* (b) *Tetrahedral sphere* (c) *Monkey pants*

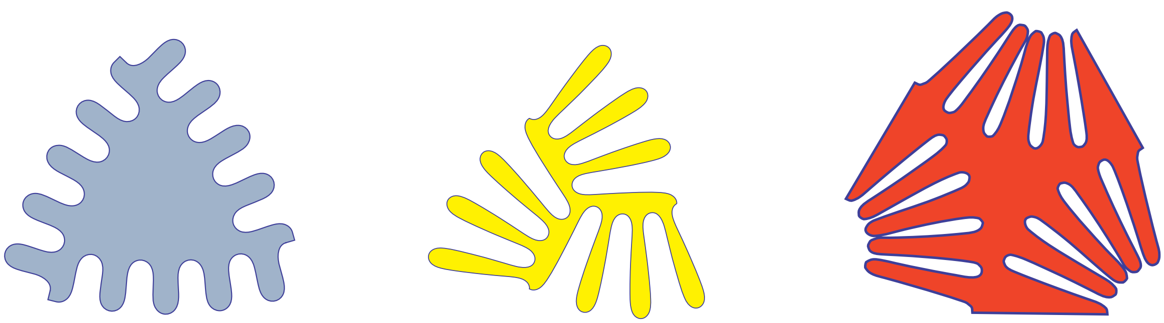


Figure 5: *Paper Models*

Construction

The patterns for these models were designed using a dynamic Mathematica notebook. This notebook has many parameters which allow one to adjust the number of fingers, finger length and shape, distribution of curvature along the side, all while keeping the total curvature along the side fixed.

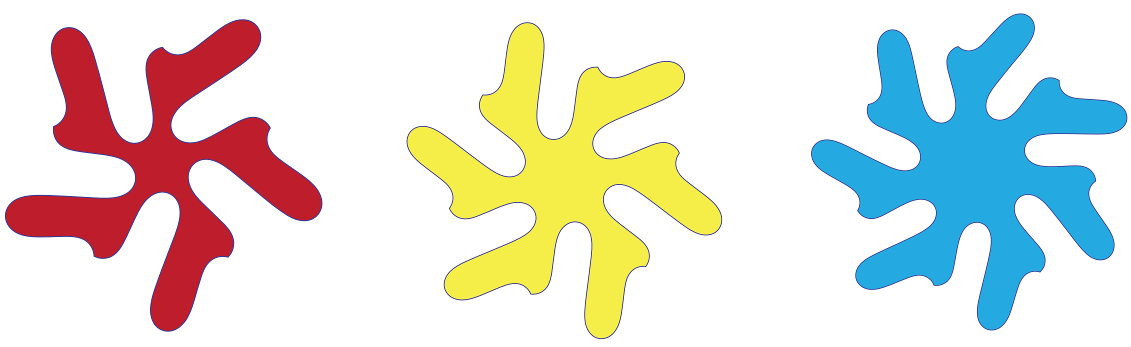
It would be very time consuming to cut the patterns by hand, and nearly impossible to get the accuracy required for satisfying finished results. For cutting paper models we used a Graftec Craft Robo cutter. George Hart describes the mechanics of using such a cutter in [1]. Once a pattern piece had been designed and graphed in Mathematica, the image was imported into Adobe Illustrator, which has a software plug-in that is compatible with the Craft Robo cutter.

After the pieces were cut from 65-100lb card stock, they were assembled using tape. With care, the job can be done almost completely by taping on the inside. The model with edges that are circle arcs, Figure 3c, is the most difficult to assemble. The curved sides require many small pieces of tape, or some snipping of longer pieces. As the curves get longer and more complicated, the dihedral angle is very close to 180°, so the taping becomes much easier. Longer pieces can be used and it is not necessary to snip the tape to accommodate for the curving. Construction does not take much skill, but does require patience.

Figure 6: *Octahedral sphere II*

We share one last paper model in Figure 6. It is also based on the octahedron, made from triangular pattern pieces with total edge bending equal to 30°. From glancing at the pieces, one would guess the resulting model would be a fairly round sphere; the area at the center of the pattern piece is quite small. However, due to the shape of the piece at the corners, which have a relatively large concentration of area, the vertices became overly flat when assembled. The resulting figure is more of a cube (the dual of the octahedron) than a round sphere. Though this example was somewhat of a failure in achieving roundness, the model is interesting and illustrates behavior that can occur when distributing curvature along irregular seams.

Zippergons



(a) 120° *pentagon* (b) 120° *hexagon* (c) 120° *heptagon*

Figure 7: *Three per vertex system*

After making pieces for constant Gaussian curvature surfaces of both positive and negative curvature, we realized with a little care we could design a system of shapes that would match each other. For example, to make a three pieces per vertex system, the polygons should have 120° vertices with edges bending as given in Table 1. Pieces designed according to this data can be found in Figure 7. Assembling 12 of these pentagons will create a smoothed dodecahedron. When hexagons are taped to hexagons, they create a flat plane, or perhaps a cylinder. By using the heptagons, one can create a smooth model of the hyperbolic plane. The lengths of the curves that make up the sides of each of the shapes are the same, so different style pieces can be taped to each other to create surfaces of mixed curvature. Theoretically, by mixing and matching enough pieces appropriately, one could make a surface in the shape of, say, a duck.

polygon	pre-bent angle	bending per side	bending per polygon	type of curvature
pentagon	$108°$	$12°$	$60°$	positive
hexagon	$120°$	0	0	0
heptagon	$128\frac{4}{7}°$	$-8\frac{4}{7}°$	$-60°$	negative

Table 1: *Bending data for* $120°$ *degree system*

As we experimented with these pieces we soon decided that the paper and tape method we had been using was not practical for building more complicated surfaces. Building even a simple torus turned out to be challenging. One constraint comes from the following theorem: Any smooth surface with no boundary embedded in three dimensional space must have some regions with positive curvature. Moreover, the total curvature in these regions must be at least 4π. Therefore, if we were to use the above pieces to build a torus, we would need to use at least 12 pentagons. The Gauss-Bonnet theorem tells us the total curvature of any smooth torus is zero, so for every pentagon we use, we must also use a heptagon. (The actual number of pieces used in constructions of surfaces without boundary tends to be much higher than the minimum dictated by Gauss-Bonnet.) Taping is tedious, and makes it impossible to change one's mind about a particular design mid construction, so we went looking for another material to work with.

One promising material was craft foam. This inexpensive, easily available material is sold in sheets at many craft stores. The nominal thickness is about 2mm, but there is some variation. After much experimentation, we were able to cut the foam using the Graphtec cutter, using special-ordered blades. We set out to design pieces that would work well with this new material. Most of our experimentation was done with a set of $90°$ polygons. Bending data for three polygons can be found in Table 2.

(a) *Torus*

(b) *Vase*

Figure 8

Figure 8 shows two creations made with an early version of what came to be called Zippergons. At this stage, it was possible to get a weak gripping along the seams in the case of negative curvature, where the teeth come together, by slightly increasing the bulbs of the teeth. However, due to the nature of positive curvature, the teeth along the sides of the triangles spread apart, requiring some sort of adhesive tape. These models were constructed using artist tape, which can be removed and reused. This allowed for more flexibility in the design. These constructions have turned out to be quite durable.

By the previous discussion, constructing a torus out of the right-angled pieces requires at least 8 triangles and 8 pentagons. The torus pictured in Figure 8a has 10 red triangles, 10 blue pentagons, and 20 yellow squares. The vase in Figure 8b was formed by inserting white squares in an octahedron to separate its 8 triangles by knights moves, and then then replacing the four top squares with a tube made from 8 pentagons.

polygon	pre-bent angle	bending per side	bending per polygon	type of curvature
triangle	60°	30°	90°	positive
square	90°	0	0	0
pentagon	108°	-18°	-90°	negative

Table 2: *Bending data for* 90° *degree system*

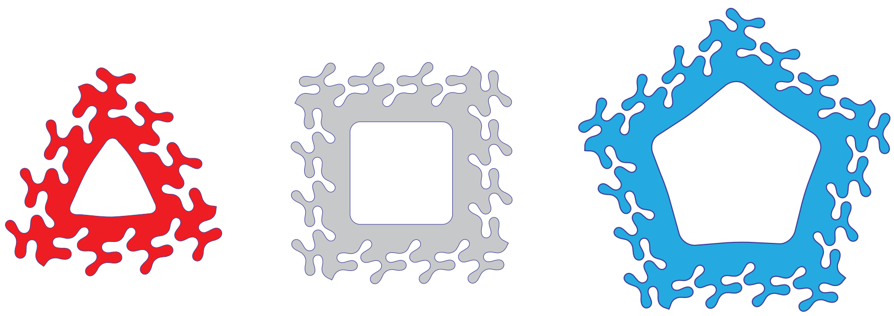


Figure 9: *Improved Zippergons*

When we tried out our models with groups of mathematicians, they attracted a lot of interest, but we found that few people had the patience to build them. The necessity of the tape made it difficult to experiment and rearrange the pieces. We then began a long process of trying to develop interlocking seams that would hold together without tape. Many more parameters were added to the mathematica file in which pieces were designed. These parameters had names such as bulge, grip, microgrip, skew, microskew, microbumping, bumpspertooth, and so on. See Figure 9 for examples of right-angled pattern pieces which require no tape.

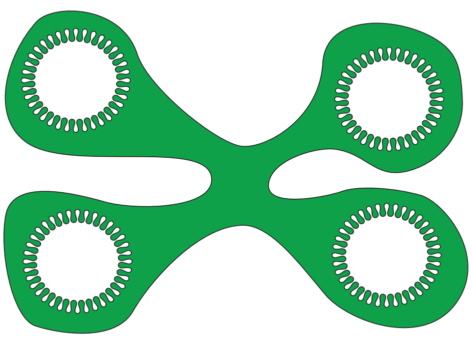
(a) *Cuboctahedron* (b) *Icosadodecahedron* (c) *Negative curvature*

Figure 10: *Tapeless Zippergon constructions*

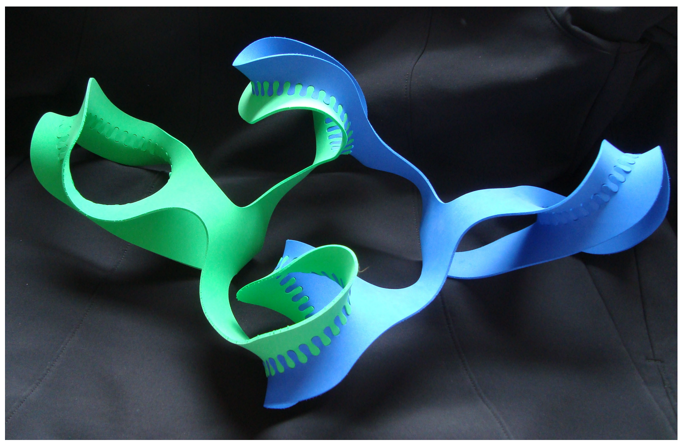
The models in Figure 10 were constructed from just foam. To assemble these pieces, one tucks the teeth snugly into the appropriate notches. To get the best grip, it is often necessary to go back over a seam and massage the pieces further into place. Recall from Table 2 that the magnitude of curving on an edge of a triangle is greater than that of a pentagon, so a triangle-pentagon pairing gives a slightly convex shape, as is seen in the icosadodecahedron. Similarly, the pentagon-square construction has a negatively curved saddle shape, though not as dramatic as the curving in a pentagon-pentagon pattern. If you get tired of having a cuboctahedron sitting around, pulling in a perpendicular direction to the seams gives immediate deconstruction. The eight triangles could then be reused to make an octahedron.

Alien

Along the way to developing no-tape versions of the Zippergons, we developed several patterns for surfaces that are more complicated than spheres and tori. One example is the Alien. The seams of this piece are more stable than perhaps any other of our designs. The unzipped alien is equivalent to a sphere with 5 disks removed. (The outer boundary of the pattern piece corresponds to the fifth hole.) The arms can be joined in a number of ways, and rotating the hands when joining gives even more possibilities to the shape of a zipped up Alien. If you join pairs of hands in one Alien, the surface you get is a two-holed torus with a disk removed. Figure 11b shows an example of two aliens that have been joined together. What surface is it?



(a) *Alien pattern*



(b) *Pair of Aliens*

Figure 11: *Alien*

Note

The designs illustrated are for academic use only. Commercial use without written permission of the authors is prohibited. The first author would like to thank the UUP for the Drescher award which allowed her to visit Cornell University in the Spring of 2010, at which time this project began. She would also like to thank the Mathematics Department at Cornell for their hospitality during this semester.

References

[1] George Hart. Modular-Kirigami. *Proceedings of Bridges Donestonia, San Sebastian Spain*, pages 1–8, 2007.