# 18.510: INTRODUCTION TO MATHEMATICAL LOGIC AND SET THEORY, FALL 08: LECTURE 1

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## 1. Propositional Calculus: Exposition

Consider variables $p$, $q$, $r$. We think of them as elementary propositions. To each of them we can assign a truth value: *true* (denoted by 1) or *false* (0). Consider the connectives: contradiction denoted by $\perp$ (which stands on its own), negation denoted by $\neg$ (not), which is placed in front of a formula, and connectives that are placed between two formulas: disjunction denoted by $\vee$ (or), conjunction denoted by $\wedge$ (and), implication denoted by $\to$ (if-then). According to our convention, the truth-value of $\perp$ is 0. The truth value of $\neg p$ is 0 if the truth value of $p$ is 1, and 1 if the truth value of $p$ is 0. The truth value of a proposition compounded by "or" depends on the truth value of its constituents as follows:

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

The truth table for "and" is:

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

The truth table for "if-then" is:

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

We can compound other propositions using the logical connectives and the variables, for example, $\neg((\neg p) \vee (\neg q))$. The truth table of $\neg((\neg p) \vee (\neg q))$ is obtained from the above truth tables:

| $p$ | $q$ | $\neg p$ | $\neg q$ | $(\neg p) \vee (\neg q)$ | $\neg((\neg p) \vee (\neg q))$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 | 1 |

1.1. *Remark.* Notice that the connectives $\neg$ (not), $\vee$ (or) and $\wedge$ (and) can be defined using only the connectives $\perp$ and $\rightarrow$.

(1) The truth table for $p \rightarrow \perp$ is

| $p$ | $\perp$ | $p \rightarrow \perp$ |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0 | 0 |

which is the same as the truth table of $\neg p$, for all $p$.

(2) The truth table for $((p \rightarrow \perp) \rightarrow q)$, i.e., $(\neg p \rightarrow q)$ is

| $p$ | $q$ | $\neg p$ | $q$ | $(\neg p \rightarrow q)$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 |

which is the same as the truth table of $p \vee q$ for all $p, q$.

(3) The truth table for $\neg(\neg p \vee \neg q)$ (which, by parts (1) and (2), can be defined using $\perp$ and $\rightarrow$), is the same as the truth table for $p \wedge q$ for all $p, q$ (as we saw above). The equivalence between $p \wedge q$ and $\neg(\neg p \vee \neg q)$ is called De-Morgan Law.

1.2. *Exercise.* Show that the truth tables of $\rightarrow$, $\wedge$ and $\perp$ can be obtained as truth tables of propositions compounded by $\vee$ and $\neg$.

**Propositional Calculus: Syntax.** We now move to formal notations. Let $\mathcal{A}$ be an *alphabet* consisting of the following symbols:

(1) $\perp$ (contradiction) (0-ary connective);
(2) $\rightarrow$ (if-then) (binary connective);
(3) ), ( (parentheses: closing parenthesis and opening parenthesis);
(4) $p_1, p_2, \ldots$ (propositional variables).

Items (1)-(3) are called the *logical symbols*. We assume that the set $P$ of propositional variables is not empty. We also assume that the logical symbols are not elements of $P$.

1.3. *Remark.* When the choice of a particular variable is not important, we will often not specify the choice and use $p$, $q$, $r$ instead.

A *propositional formula* over $\mathcal{A}$ is defined by the following definition.

- $\perp$ is a propositional formula;
- every propositional variable $p_i$ is a propositional formula;
- if $\alpha$ and $\beta$ are propositional formulas then $(\alpha \to \beta)$ is also a propositional formula;
- no other string over $\mathcal{A}$ is a propositional formula.

(A propositional formula is obtained by applying the first three rules finitely many times.) For example, $(p_0 \to (p_1 \to \perp))$ is a propositional formula. Notice that there are strings over $\mathcal{A}$ that are not propositional formulas, e.g., $\to p$, $)\perp$, etc. We sometimes say formulas or propositions instead of propositional formulas.

1.4. *Remark.* The above definition of a propositional formula is a "definition from above", or *inductive*: we are defining the smallest subset of a fixed set $E$ that includes a given subset and is closed under certain operations defined on $E$. We have an equivalent definition "from below": this consists in constructing the set one level at a time: the given subset is the lowest level and the elements of level $n+1$ are defined to be the images under the given operations of the elements from the lower levels. In our case, the given set is the union of $\perp$ and the set of propositional variables, and the operation is getting $(\alpha \to \beta)$ out of two formulas $\alpha$ and $\beta$. For more details, see Theorem 1.3 (p.10) in the Cori-Lascar textbook.

The symbols $), ($ play an important role; thanks to them there is a unique way to obtain a proposition.

1.5. **Theorem** (Unique Decomposition)**.** *For any propositional formula $\phi$, one and only one of the following three cases can arise:*

(1) $\phi = \perp$.
(2) $\phi$ *is a propositional variable.*
(3) *There is a unique pair of formulas* $(\phi_1, \phi_2)$ *such that* $\phi = (\phi_1 \to \phi_2)$.

Proof: guided exercise in Problem Set 1.

1.6. *Remark.* By Remark 1.1, for propositional formulas $\alpha$, $\beta$, the propositions $\neg\alpha$, $(\alpha \vee \beta)$, and $(\alpha \wedge \beta)$ can be written as propositional formulas. We will use the notations $\neg\alpha$, $(\alpha\vee\beta)$, $(\alpha\wedge\beta)$ as abbreviations for the corresponding propositional formulas.

For a propositional formula $\alpha$, let $\mathrm{pvar}(\alpha)$ be the set of propositional variables occurring in $\alpha$,

$$\mathrm{pvar}(\alpha) = \{p \mid p \text{ occurs in } \alpha\}.$$