# 18.510: INTRODUCTION TO MATHEMATICAL LOGIC AND SET THEORY, FALL 08

LIAT KESSLER

## 1. Propositional Calculus: Exposition

**Propositional Calculus: Semantics.** An *assignment* is a map $b$ from the set of propositional variables $\{p_1, p_2, \ldots\}$ to $\{0, 1\}$ that assigns truth value: 0 if false, 1 if true. The truth value $b(\alpha)$ of a propositional formula $\alpha$ under the assignment $b$ is defined recursively, (by recursion on the construction of the formula), as follows.

- $b(\perp) = 0$;
- $b(p_i)$ is given for all propositional variables $p_i$;
- If $\gamma$ is of the form $(\alpha \to \beta)$, then, (by the truth table of $\to$), $b(\gamma) = 0$ if $b(\alpha) = 1$ and $b(\beta) = 0$; otherwise $b(\gamma) = 1$.

(Well defined because of Theorem **??**.)

The truth value $b(\alpha)$ depends only on the assignment of the propositional variables occurring in the formula $\alpha$.

1.1. **Lemma** (Coincidence Lemma of Propositional Logic.)**.** *Let $\alpha$ be a propositional formula and let $b$ and $b'$ be assignments with $b(p) = b'(p)$ for all $p \in \mathrm{pvar}(\alpha)$. Then $b(\alpha) = b'(\alpha)$.*

The proof is by (an easy) induction on the formulas (check).

If $b(\alpha)$ is 1 we say that $b$ is a *model* of $\alpha$, or that it *satisfies* $\alpha$. The assignment $b$ is a *model* of a set $\Delta$ of propositional formulas if $b$ is a model of each formula in $\Delta$.

We assume that the set of propositional variables in the alphabet is countable. We say that a set $S$ is *countable* if there is a surjective map $\alpha$ from the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ to $S$. We can then represent $S$ as $\{\alpha(n) \,|\, n \in \mathbb{N}\}$, or (if we write the arguments as indices, as $\{\alpha_n \,|\, n \in \mathbb{N}\}$.

1.2. **Theorem.** *The set $\Phi$ of all propositional formulas (in variables $p_1, p_2, \ldots$) is countable.*

Set

$$\Phi_\ell^k$$

to be the set of all propositional formulas $\alpha$ for which $\mathrm{pvar}(\alpha) \subseteq \{p_1, \ldots, p_k\}$ and the symbol $\rightarrow$ appears $\ell$ times or less.

1.3. **Lemma.**

$$|\Phi^k_\ell| \leq (k+1)^{2^\ell} \cdot (\ell+1)^{2^\ell}$$

*Proof.* By induction on $\ell$ (simultaneously for all $k$). For $\ell = 0$, $\Phi^k_0 = \{\bot, p_1, \ldots, p_k\}$, hence

$$|\Phi^k_0| = k + 1 = (K+1)^{2^0} \cdot (0+1)^{2^0}.$$

Induction step: If $\phi \in \Phi^k_{\ell+1}$, then either $\phi \in \Phi^k_0$ or $\phi = (\alpha \rightarrow \beta)$ for $\alpha$, $\beta \in \Phi^k_l$. The number of possible $\alpha$ ($\beta$) is $|\Phi^k_\ell|$, hence the number of possible $(\alpha \rightarrow \beta)$ is $|\Phi^k_\ell|^2$. Therefore

$$|\Phi^k_{l+1}| \leq |\Phi^k_l|^2 + (k+1) \leq ((k+1)^{2^\ell} \cdot (\ell+1)^{2^\ell})^2 + (k+1) \leq (k+1)^{2^{\ell+1}} \cdot (\ell+2)^{2^{\ell+1}}.$$

The second inequality is by the induction assumption. □

*Proof of Theorem 1.2.* We notice that for the set $\Phi$ of all propositional formulas,

$$\Phi = \cup_{k=1}^{\infty} \Phi^k_k.$$

By Lemma 1.3, $\Phi^k_\ell$ is finite for all $\ell$, $k$. In particular

$$\Phi^k_k \setminus \Phi^{k-1}_{k-1}$$

is finite. Hence we can write $\Phi^1_1 = \{\phi_1, \ldots, \phi_{n_1}\}$, $\Phi^2_2 \setminus \Phi^1_1 = \{\phi_{n_1+1}, \ldots, \phi_{n_1+n_2}\}$, and so on. We get that $\Phi$ is countable: $\Phi = \{\phi_1, \ldots, \phi_{n+1}, \ldots\}$. □

We denote by $\mathcal{B}$ the set of all assignments. Assume that the alphabet is countable and not finite.

1.4. **Theorem.** *The collection $\mathcal{B}$ of assignments, i.e., the maps from the set of propositional formulas to $\{0, 1\}$ is not countable.*

The proof is by a diagonal argument due to Cantor.

*Proof.* Assume that $\mathcal{B}$ is countable, i.e., $\mathcal{B} = \{b_1, b_2, \ldots\}$. Define an assignment $b^*$ as follows: for the variable $p_i$,

$$b^*(p_i) = 1 - b_i(p_i),$$

i.e., $b^*(p_i) = 0$ if $b_i(p_i) = 1$, and $b^*(p_i) = 1$ if $b_i(p_i) = 0$.

Then for all $i \in \mathbb{N}$, we get $b^* \neq b_i$, since they do not agree for $p_i$, in contradiction with the assumption. □

**Logical consequence.** Let $\Delta$ be a set of propositional formulas, and let $\alpha$, $\beta$ be propositional formulas. We say that $\alpha$ is a (logical) *consequence* of $\Delta$, and write $\Delta \models \alpha$ if every model of $\Delta$ is a model of $\alpha$. When $\Delta$ consists of a single formula $\phi$ we sometimes abbreviate and write $\phi \models \alpha$ for $\{\phi\} \models \alpha$.

We say that $\alpha$ is a *tautology*, and write $\models \alpha$, (or $\emptyset \models \alpha$), if $\alpha$ is true (i.e., has truth value 1) under all assignments.

We say that $\Delta$ is *satisfiable*, and write $\operatorname{Sat} \Delta$ if there is an assignment which is a model of $\Delta$. We say that $\alpha$ is *satisfiable* if $\operatorname{Sat}\{\alpha\}$.

We say that $\alpha$ and $\beta$ are logically equivalent if $\models (\alpha \to \beta) \wedge (\beta \to \alpha)$.

1.5. *Example.* For every propositional formula $\alpha$, the propositional formula

$$((\neg(\neg\alpha)) \to \alpha) \tag{1}$$

is a tautology. Indeed, for every assignment $b$, if $b((\alpha \to \bot) \to \bot) = 1$ then, by definition (of an assignment), $b(\alpha \to \bot) = 0$, and hence (by definition) $b(\alpha) = 1$, therefore by definition $b(\neg(\neg\alpha) \to \alpha) = 1$.