18.510: INTRODUCTION TO MATHEMATICAL LOGIC AND SET THEORY, FALL 08

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1. Propositional Calculus

Notation

- For any non-zero natural number k, if $\phi_1, \dots, \phi_k$ are formulas, we let $\phi_1 \wedge \dots \wedge \phi_k$ represent the formula $((\dots(\phi_1 \wedge \phi_2) \wedge \phi_3) \wedge \dots \wedge \phi_k)$ (which begins with $k-1$ occurrences of the open parenthesis symbol).
- If $I = \{i_1, \dots, i_k\}$ is a non-empty set of indices, and $\phi_1, \dots, \phi_k$ are formulas, the formula $\phi_{i_1} \wedge \dots \wedge \phi_{i_k}$ will also be written $\wedge_{j \in I} \phi_j$. (We ignore the ambiguity relating to the indices in the set I since we are concerned with semantics, and we have the commutativity of conjunction up to logical equivalence.)
- We make analogous conventions for disjunction.

Propositional connectives. For $n \in \mathbb{N}$, we denote by $\mathcal{B}_n$ the set of assignments of 0/1 value to the propositional variables $p_1, \dots, p_n$. Notice that $\mathcal{B}_n$ can be identified with $\{0,1\}^n$ hence $|\mathcal{B}_n| = 2^n$.

A *k-place Boolean function*, sometimes called a *k-place propositional connective*, is a function from $\mathcal{B}_k$ into $\{0,1\}$; equivalently, a function from $\{0,1\}^k$ into $\{0,1\}$. We permit 0 and 1 themselves as 0-place Boolean functions. For each $n \in \mathbb{N}$ there are 2^{2^n} n-place Boolean functions. From a propositional formula α with $\operatorname{pvar} \alpha \subseteq \{p_1, \dots, p_n\}$ we define an n-place Boolean function B^n_α, the Boolean function *realized* by α, by

$$B^n_\alpha(b) = b(\alpha), \text{ for } b \in \mathcal{B}_n.$$

1.1. **Lemma.** *Let $n \in \mathbb{N}$. For every n-place Boolean function $f \colon \mathcal{B}_n \to \{0,1\}$, there is a formula ϕ with $\operatorname{pvar}(\phi) \subseteq \{p_1, \dots, p_n\}$ such that f is realized by ϕ, i.e.,*

$$f(b) = b(\phi), \quad \textit{for all } b \in \mathcal{B}_n.$$

Of course the formula ϕ that realizes f is not unique; any logically equivalent formula will also realize the same function.

Proof. Step 1. Special case: there is exactly one $b^* \in \mathcal{B}_n$ for which $f(b^*) = 1$; for $\mathcal{B}_n \ni b \neq b^*$, $f(b) = 0$. Set

$$q_i = \begin{cases} p_i & \text{if } b^*(p_i) = 1; \\ \neg p_i & \text{if } b^*(p_i) = 0. \end{cases}$$

Define

$$\phi = \wedge_{i=1}^n q_i.$$

We claim that for all $b \in \mathcal{B}_n$, $f(b) = b(\phi)$. Indeed, for b^*, we have $f(b^*) = 1 = b^*(\phi)$ by definition of ϕ (for all q_i, $b^*(q_i) = 1$, hence $b^*(\wedge_{i=1}^n q_i) = 1$). If $b \neq b^*$, there is (at least) one p_i such that $b(p_i) \neq b^*(p_i)$, hence $b(p_i) = 0$, therefore $b(\phi) = b(\wedge_{i=1}^n q_i) = 0 = f(b)$.

Step 2. Special case: for all $b \in \mathcal{B}_n$, $f(b) = 0$. Then for

$$\phi = \perp,$$

or

$$\phi = (p_1 \wedge \neg p_1)$$

(if $n \geq 1$), $b(\phi) = 0 = f(b)$ for all $b \in \mathcal{B}_n$.

Step 3. General case. First enumerate the elements of $\mathcal{B}_n$ from 1 to 2^n to get a list $(b_1, b_2, \ldots, b_{2^n})$. Given $f\colon \mathcal{B}_n \to \{0,1\}$, for $1 \leq j \leq 2^n$, set $f_j\colon \mathcal{B}_n \to \{0,1\}$ as follows:

$$f_j(b) = \begin{cases} f(b) & \text{if } b = b_j; \\ 0 & \text{otherwise.} \end{cases}$$

Apply the special cases to find ϕ_j such that $f_j(b) = b(\phi_j)$ for all $b \in \mathcal{B}_n$. Now set

$$\phi = \vee_{j=1}^{2^n} \phi_j.$$

For b_k, the value $b_k(\phi) = \vee_{j=1}^{2^n} b_k(\phi_j)$ is 1 if $1 = b_k(\phi_k) = f_k(b_k) = f(b_k)$, and 0 if $0 = b_k(\phi_k) = f_k(b_k) = f(b_k)$ (notice that for $j \neq k$, $b_k(\phi_j) = 0$). In other words, $b(\phi) = 1$ if and only if there is $1 \leq k \leq 2^n$ such that $f_k(b) = 1$ if and only if $f(b) = 1$.

Notice that we can omit the ϕ_j's that equal $\perp$ (unless they all do). □

1.2. A formula ϕ is in a *disjunctive normal form* (DNF) if and only if there exist

- an integer $m \geq 1$,
- integers $k_1, \ldots, k_m \geq 1$,
- for every $i \in \{1, \ldots, m\}$, k_i propositional variables $p_{i_1}, \ldots, p_{i_k}$ and k_i elements $\delta_{i_1}, \ldots, \delta_{i_{k_i}}$ in $\{0,1\}$ such that

$$\phi = \vee_{i=1}^m (\delta_{i_1} p_{i_1} \wedge \ldots \wedge \delta_{i_{k_i}} p_{i_{k_i}}),$$

where by $1p$ we mean p and by $0p$ we mean $\neg p$.

A formula ϕ with $\text{pvar}(\phi) \subseteq \{p_1, \ldots, p_n\}$ is in a *canonical disjunctive normal form* (CDNF) if and only if there exists a nonempty subset X of $\{0,1\}^n$ such that

$$\phi = \vee_{(\delta_1,\ldots,\delta_n)\in X}(\wedge_{i=1}^n \delta_i p_i).$$

(Notice that CDNF is a special case of DNF where each k_i is equal to n, for each $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,n\}$, $p_{i_j} = p_j$, and the m n-tuples $\delta_{i_1}, \ldots, \delta_{i_{k_i}}$ are pairwise distinct; note that this forces m to be $\leq 2^n$.)

By interchanging the symbols for disjunction and conjunction in the definitions of DNF and CDNF we obtain the definition of a formula being in *conjunctive normal form* (CNF) and a formula being in *canonical conjunctive normal form* (CCNF). (A propositional formula is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of propositional variables or negated propositional variables.)

By examining the proof of Lemma 1.1 we see that given an n-place Boolean function $f\colon \mathcal{B}_n \to \{0,1\}$ distinct from the zero mapping, there is a formula ϕ in CDNF such that f is realized by ϕ. We conclude as well a uniqueness for canonical disjunctive normal forms, in the sense that two canonical disjunctive normal forms which are logically equivalent can differ only in the "order of their factors". More precisely if the formulas $\vee_{(\delta_1,\ldots,\delta_n)\in X}(\wedge_{i=1}^n \delta_i p_i)$ and $\vee_{(\epsilon_1,\ldots,\epsilon_n)\in Y}(\wedge_{i=1}^n \epsilon_i p_i)$ are logically equivalent, then the subsets X and Y of $\{0,1\}^n$ are identical.

The advantages of formulas in DNF stem from the fact that they explicitly list the truth assignments satisfying the formula: define an assignment $b_{\delta_1,\ldots,\delta_n} \in B_n$ by $b_{\delta_1,\ldots,\delta_n}(p_i) = \delta_i$; the formula $\vee_{(\delta_1,\ldots,\delta_n)\in X}(\wedge_{i=1}^n \delta_i p_i)$ is satisfied by the truth assignments $b_{\delta_1,\ldots,\delta_n}$ for which $(\delta_1,\ldots,\delta_n) \in X$ and only by these.

The analogous facts are true for conjunctive normal forms.

1.3. **Theorem** (Theorem on the Disjunctive Normal Form)**.** *Every propositional formula is logically equivalent to a formula in disjunctive normal form. Every propositional formula that is not logically equivalent to* $\perp$ *is logically equivalent to a unique formula in* CDNF.

Uniqueness here is understood as up to the order of the factors. An analogous theorem for CNF is also true. Every formula that is not a tautology is logically equivalent to a unique formula in CCNF.

Proof. Let α be a propositional formula. Let n be such that $\text{pvar}(\alpha) \subseteq \{p_1,\ldots,p_n\}$. Define $f\colon \mathcal{B}_n \to \{0,1\}$ by $f(b) = b(\alpha)$. By Lemma 1.1, there is a DNF formula ϕ with $\text{pvar}(\phi) \subseteq \{p_1,\ldots,p_n\}$ such that $f(b) =$

$b(\phi)$, for all $b \in \mathcal{B}_n$; If f is not the zero function, ϕ in CDNF. Since α and ϕ agree for every 0/1-assignment to the propositional variables occurring in them, they are logically equivalent. □

Because every k-th place Boolean function, for $k \geq 1$, can be realized by a propositional formula using only the connective symbols $\{\vee, \wedge, \neg\}$, we say that the set $\{\vee, \wedge, \neg\}$ is *complete*. The completeness of $\{\vee, \wedge, \neg\}$ can be improved using De Morgan's Law $\models \beta \vee \gamma \leftrightarrow \neg(\neg\beta \wedge \neg\gamma)$ to get that both $\{\vee, \neg\}$ and $\{\wedge, \neg\}$ are complete. By our definition of propositional formulas (and Remark 1.1 in Lecture 1), the set $\{\perp, \rightarrow\}$ is complete. In fact, because with this set we can realize even the two 0-place Boolean functions, it is *supercomplete*. Once we have a complete set of connectives we know that any formula is logically equivalent to one all of whose connectives are in that set. (For given any formula ϕ, we can make α using those connectives and realizing B_ϕ. Then α and ϕ are logically equivalent.

For a propositional formula ϕ, given an assignment b of truth values to the propositional variables, it is "easy" to check whether b is a model of ϕ: the number of required steps is a linear function of $n = |\operatorname{pvar}(\phi)|$. It is also possible to check whether there is a model of ϕ, i.e., whether $\operatorname{Sat}\phi$, by finding the truth table of ϕ. Notice that this requires $\sim 2^n$ steps. This problem is NP-complete.