18.510: INTRODUCTION TO MATHEMATICAL LOGIC AND SET THEORY, FALL 08

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1. Propositional Calculus

Deduction Lemma and Modus Ponens in Semantics.

1.1. **Lemma** (Deduction Lemma in Semantics). *Let Γ be a set of propositional formulas, and let ψ, ϕ be propositional formulas.*

If $\Gamma \cup \{\psi\} \models \phi$ then $\Gamma \models (\psi \to \phi)$.

Proof. Let b be a model of Γ. If $b(\psi) = 0$ then, in particular, $b(\psi \to \phi) = 1$. If $b(\psi) = 1$ then, b is a model of $\Gamma \cup \{\psi\}$, hence (by assumption) b is a model of ϕ, i.e., $b(\phi) = 1$; thus $b(\psi \to \phi) = 1$. $\square$

1.2. **Corollary.** *For all propositional formulas α, β, the formula*

$$(\alpha \to (\beta \to \alpha)) \tag{1}$$

is a tautology.

Proof. It is clear that $\{\alpha, \beta\} \models \alpha$. Hence by the Deduction Lemma, $\{\alpha\} \models (\beta \to \alpha)$. Hence, by applying the Deduction Lemma again, $\models (\alpha \to (\beta \to \alpha))$. $\square$

The "opposite" of the Deduction Lemma is also true.

1.3. **Lemma** (Modus Ponens (mp) in Semantics). *For propositional formulas α, β, we have $\{(\alpha \to \beta), \alpha\} \models \beta$.*

Proof. Assume not, i.e., there is an assignment b such that $b(\alpha \to \beta) = 1$ and $b(\alpha) = 1$ but $b(\beta) = 0$. Since, $b(\alpha) = 1$ and $b(\beta) = 0$ we get that $b(\alpha \to \beta) = 0$ in contradiction with our assumption. $\square$

1.4. **Lemma.** *Let Γ be a set of propositional formulas, and let ψ, ϕ be propositional formulas. If $\Gamma \models \phi$ and $\Gamma \cup \{\phi\} \models \psi$ then $\Gamma \models \psi$.*

Proof: exercise in PS2.

As a result of the logical Deduction Lemma and Modus Ponens, we get the following corollary.

1.5. **Corollary.** *For all propositional formulas α, β, γ, the formula*

$$((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))) \tag{2}$$

is a tautology.

Proof. Denote

$$\Delta = \{(\alpha \to (\beta \to \gamma)), (\alpha \to \beta), \alpha\}.$$

Then by mp, since $\{(\alpha \to \beta), \alpha\} \subset \Delta$, we have

$$\Delta \models \beta,$$

and since $\{(\alpha \to (\beta \to \gamma), \alpha\} \subset \Delta$, we have

$$\Delta \models (\beta \to \gamma).$$

Hence, by Lemma 1.4 and mp, we get

$$\Delta \models \gamma.$$

Therefore, by the Deduction Lemma,

$$\{(\alpha \to (\beta \to \gamma)), (\alpha \to \beta)\} \models (\alpha \to \gamma).$$

Applying the Deduction Lemma again, we get

$$\{(\alpha \to (\beta \to \gamma))\} \models ((\alpha \to \beta) \to (\alpha \to \gamma)).$$

Therefore (by the same lemma),

$$\emptyset \models ((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))).$$

□

Recall that in a previous lecture we saw that

$$(\neg\neg\alpha \to \alpha) \tag{3}$$

is a tautology (for every formula α). A propositional formula of the form (1), or (2), or (3) will be called an *axiom*. We have shown that every axiom is a tautology. Notice that there are infinitely many axioms, but also infinitely many tautologies that are not axioms, e.g., $\alpha \to \alpha$.

Propositional Logic: a Sequent Calculus. We say that a sequence $\bar{\beta}$ of finitely many formulas, $\bar{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, is a *formal proof* of a formula ϕ from a set Γ of formulas,if $\beta_n = \phi$ and for all i, either

- β_i is an axiom;
- $\beta_i \in \Gamma$;
- (mp) $\beta_i = \gamma$, and there are j, $k < i$ such that $\beta_j = (\alpha \to \gamma)$ and $\beta_k = \alpha$.

A formula ϕ is *formally provable* or *derivable* from a set Γ of formulas, written $\Gamma \vdash \phi$, if there exists a formal proof $\bar{\beta} = (\beta_1, \ldots, \beta_n)$ of ϕ from Γ.

1.6. *Example.*

$$\{p\} \vdash ((\neg q) \rightarrow p).$$

To see this, write the formal proof $\bar{\beta} = (\beta_1, \beta_2, \beta_3)$, where

$\beta_1 = (p \rightarrow ((\neg q) \rightarrow p))$	axiom of the type (2)
$\beta_2 = p$	element of $\{p\}$
$\beta_3 = ((\neg q) \rightarrow p)$	mp, $j = 1$, $k = 2$

1.7. **Lemma.** *For all Γ and ϕ, if $\Gamma \vdash \phi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \phi$.*

Proof. If $\Gamma \vdash \phi$, then there is a formal proof $\bar{\beta} = (\beta_1, \ldots, \beta_n)$ of ϕ from Γ. Let Γ_0 be the set of elements of $\bar{\beta}$ that are in Γ. Then $|\Gamma_0| \leq n$, and $\Gamma_0 \vdash \phi$ (with the same formal proof $\bar{\beta}$ and the same justifications). □

We say that set of formulas Γ is *consistent*, if there is no formal proof of $\perp$ from Γ.

1.8. **Corollary.** *For a set of formulas Γ, if every finite subset of Γ is consistent then so is Γ.*

Proof. Assume that Γ is not consistent, i.e., $\Gamma \vdash \perp$, then by Lemma 1.7, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \perp$, i.e., there is a finite subset of Γ that is not consistent. □