18.510: INTRODUCTION TO MATHEMATICAL LOGIC AND SET THEORY, FALL 08

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1. Propositional Calculus

Propositional Logic: a Sequent Calculus. Recall: we say that a sequence $\bar{\beta}$ of finitely many formulas, $\bar{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, is a *formal proof* of a formula ϕ from a set Γ of formulas,if $\beta_n = \phi$ and for all i, either

- β_i is an axiom;
- $\beta_i \in \Gamma$;
- (mp) $\beta_i = \gamma$, and there are j, $k < i$ such that $\beta_j = (\alpha \to \gamma)$ and $\beta_k = \alpha$.

A formula ϕ is *formally provable* or *derivable* from a set Γ of formulas, written $\Gamma \vdash \phi$, if there exists a formal proof $\bar{\beta} = (\beta_1, \ldots, \beta_n)$ of ϕ from Γ.

1.1. **Lemma.** *Let ψ be a formula. Then $\emptyset \vdash (\psi \to \psi)$.*

Proof: exercise in PS2.

1.2. **Lemma.** *Let Γ be a set of formulas, and let ψ be a formula. If ϕ is an axiom, then $\Gamma \vdash (\psi \to \phi)$.*

Proof. The sequence $\bar{\beta}$:

$\beta_1 = (\phi \to (\psi \to \phi))$	axiom
$\beta_2 = \phi$	axiom
$\beta_3 = (\psi \to \phi)$	mp, $j = 1$, $k = 2$

is a formal proof of $(\psi \to \phi)$ from Γ. □

1.3. **Lemma.** *Let Γ be a set of formulas, and let ψ be a formula. If $\phi \in \Gamma$, then $\Gamma \vdash (\psi \to \phi)$.*

Proof. The same $\bar{\beta}$ as in the proof of Lemma 1.2 will work, but the justification for β_2 now is that $\phi \in \Gamma$. □

1.4. **Lemma** (Deduction Lemma)**.** *Let Γ be a set of propositional formulas, and let ψ, ϕ be propositional formulas. If $\Gamma \cup \{\psi\} \vdash \phi$ then $\Gamma \vdash (\psi \to \phi)$.*

Proof. Assume that there is a proof $\bar{\beta} = (\beta_1, \ldots, \beta_\ell)$ of ϕ from $\Gamma \cup \{\psi\}$. We will show by induction on $i = 1, 2, \ldots, \ell$ that

$$\Gamma \vdash (\psi \to \beta_i) \tag{1}$$

for all i. For $i = \ell$, we get $\Gamma \vdash (\psi \to \beta_\ell)$, i.e., $\Gamma \vdash (\psi \to \phi)$.

- If β_i is an axiom, then (1) follows from Lemma 1.2.
- If $\beta_i \in \Gamma$, then (1) follows from Lemma 1.3.
- If $\beta_i = \psi$, then (1) follows from Lemma 1.1.
- (mp) If there are j, $k < i$ such that

  $$\beta_j = (\alpha \to \gamma),$$

  $$\beta_k = \alpha,$$

  and

  $$\beta_i = \gamma,$$

  then by the induction assumption

  $$\Gamma \vdash (\psi \to (\alpha \to \gamma))$$

  and

  $$\Gamma \vdash (\psi \to \alpha).$$

  Let

  $$(\delta_1, \ldots, \delta_m)$$

  be a proof of $(\psi \to (\alpha \to \gamma))$ from Γ, and let

  $$(\epsilon_1, \ldots, \epsilon_n)$$

  be a proof of $(\psi \to \alpha)$ from Γ. Set

  $$\delta_{m+i} = \epsilon_i.$$

  Then

  δ_1

  $\vdots$

  $\delta_m = (\psi \to (\alpha \to \gamma))$

  δ_{m+1}

  $\vdots$

  $\delta_{m+n} = (\psi \to \alpha)$

  $\delta_{m+n+1} = ((\psi \to (\alpha \to \gamma)) \to ((\psi \to \alpha) \to (\psi \to \gamma))$ axiom

  $\delta_{m+n+2} = ((\psi \to \alpha) \to (\psi \to \gamma))$ mp on δ_m and δ_{m+n+1}

  $\delta_{m+n+3} = (\psi \to \gamma)$ mp on δ_{m+n} and δ_{m+n+2}

  Now, $(\delta_1, \ldots, \delta_{m+n+3})$ is a proof of $(\psi \to \beta_\ell)$ from Γ as claimed.

□

The deduction Lemma is not itself formulated with propositional calculus: it is not a theorem of propositional calculus, but a theorem about propositional calculus. In this sense, it is a meta-theorem, comparable to theorems about the soundness or completeness of propositional calculus.

1.5. **Corollary.**

$$\{(p \to q), (q \to r)\} \vdash (p \to r).$$

Proof. It follows from the Deduction Lemma with $\Gamma = \{(p \to q), (q \to r)\}$ and $\psi = p$. □

1.6. **Corollary** (Contraposition)**.** $\Gamma \cup \{\phi\} \vdash \neg\psi$ *iff* $\Gamma \cup \{\psi\} \vdash \neg\phi$.

Proof: exercise in PS2.

The Completeness Theorem.

1.7. **Theorem** (Completeness Theorem)**.** *Let* Γ *be a set of formulas, and let* ψ *be a formula. Then*

$$\Gamma \vdash \psi \text{ if and only if } \Gamma \models \psi.$$

We will prove the $\Rightarrow$ direction directly.

Proof that if $\Gamma \vdash \psi$ *then* $\Gamma \models \psi$. Let $\bar{\beta} = (\beta_1, \ldots, \beta_n)$ be a proof of ψ from Γ. We will show by induction on $i = 1, \ldots, n$ that $\Gamma \models \beta_i$ and conclude that $\Gamma \models \beta_n$ since $\beta_n = \psi$.

- If β_i is an axiom then $\models \beta_i$ (as we saw before), hence $\Gamma \models \beta_i$.
- If $\beta_i \in \Gamma$, then, by definition, $\Gamma \models \beta_i$.
- (mp) If there are $j, k < i$ such that $\beta_j = (\alpha \to \gamma)$, $\beta_k = \alpha$ and $\beta_i = \gamma$, then by induction assumption, $\Gamma \models (\alpha \to \gamma)$, and $\Gamma \models \alpha$. By Modus Ponens in Semantics, this implies $\Gamma \models \gamma$.

□