18.510: INTRODUCTION TO MATHEMATICAL LOGIC AND SET THEORY, FALL 08

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1. Propositional Calculus

The Completeness Theorem.

1.1. **Theorem** (Completeness Theorem). *Let* Γ *be s set of formulas, and let* ψ *be a formula. Then*

$$\Gamma \vdash \psi \text{ if and only if } \Gamma \models \psi.$$

We will prove the $\Rightarrow$ direction directly.

Proof that if $\Gamma \vdash \psi$ *then* $\Gamma \models \psi$. Let $\bar{\beta} = (\beta_1, \dots, \beta_n)$ be a proof of ψ from Γ. We will show by induction on $i = 1, \dots, n$ that $\Gamma \models \beta_i$ and conclude that $\Gamma \models \beta_n$ since $\beta_n = \psi$.

- If β_i is an axiom then $\models \beta_i$ (as we saw before), hence $\Gamma \models \beta_i$.
- If $\beta_i \in \Gamma$, then, by definition, $\Gamma \models \beta_i$.
- (mp) If there are $j, k < i$ such that $\beta_j = (\alpha \to \gamma)$, $\beta_k = \alpha$ and $\beta_i = \gamma$, then by induction assumption, $\Gamma \models (\alpha \to \gamma)$, and $\Gamma \models \alpha$. By mp in semantics, this implies $\Gamma \models \gamma$.

□

We will deduce the $\Leftarrow$ direction from the following theorem.

1.2. **Theorem** (Model Existence Theorem). *Every consistent set of propositional formulas is satisfiable.*

Proof that if $\Gamma \models \psi$ *then* $\Gamma \vdash \psi$*, given the Model Existence Theorem.* Since $\Gamma \models \psi$, for every model b of Γ, we have $b(\psi) = 1$ hence $b(\neg\psi) = 0$, i.e., there is no model of $\Gamma \cup \{\neg\psi\}$. Hence, by the Model Existence Theorem, $\Gamma \cup \{\neg\psi\}$ is not consistent, i.e.,

$$\Gamma \cup \{\neg\psi\} \vdash \bot.$$

By the Deduction Lemma, this implies that $\Gamma \vdash (\neg\psi \to \bot)$, i.e.,

$$\Gamma \vdash \neg\neg\psi.$$

Let

$$\bar{\beta} = (\beta_1, \dots, \beta_n)$$

be a proof of $\neg\neg\psi$ from Γ. Set

$\beta_{n+1} = (\neg\neg\psi \to \psi)$ axiom

$\beta_{n+2} = \psi$ mp on β_n and β_{n+1}.

Now $(\beta_1, \ldots, \beta_{n+2})$ is a proof of ψ from Γ.

□

As a corollary of the Completeness Theorem we get the following result.

1.3. **Corollary.** *$\vdash \alpha$ iff α is a tautology.*

This means that every tautology can be formally proven from the set of axioms (and mp), that is, we should not have included more tautologies in the set of axioms.

The Compactness Theorem.

1.4. **Theorem** (Compactness Theorem)**.**
(1) *For all Γ and ϕ, if $\Gamma \models \phi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \phi$.*
(2) *For a set of formulas Γ, if every finite subset of Γ admits a model, then so does Γ.*

1.5. *Exercise.*
(1) Show that part (2) follows directly from part (1).
(2) Show that part (1) follows directly from part (2).

Proof of part (1) given the Completeness Theorem. If $\Gamma \models \phi$, then by the Completeness Theorem, $\Gamma \vdash \phi$, hence, (see Lemma **??**), there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \phi$, therefore (by the easy direction of the Completeness Theorem) $\Gamma \models \phi$. □

Proof of part (2) given the Model Existence Theorem. Assume that every finite subset of Γ admits a model. It is enough to show that Γ is consistent, and therefore by the Model Existence Theorem, it admits a model. Assume that Γ is not consistent, then by Corollary **??**, there is a finite subset $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 \vdash \bot$, hence $\Gamma_0 \models \bot$ (by the easy direction of the completeness theorem). On the other hand, by assumption there is a model b of Γ_0; since $\Gamma_0 \models \bot$, we get that b is a model of $\bot$, i.e., $b(\bot) = 1$, in contradiction with the definition of an assignment. □

1.6. *Remark.* In the literature you will find other systems of proof ($\vdash$). In our system, there are infinitely many axioms (in three types) and one inference rule: Modus Ponens. In other systems there might be more inference rules. The main properties that are required from a system of proof are: the axioms and inference rules are correct and complete, i.e., the Completeness Theorem holds, and the finiteness of a proof (which imply the Compactness Theorem).

Proof of the Model Existence Theorem. Before we prove the Model Existence Theorem, we need to understand the following concept. We say that a set of formulas Γ *decides* a formula ϕ if either $\Gamma \vdash \phi$ or $\Gamma \vdash (\neg\phi)$.

Notice that it might be that neither $\Gamma \vdash \phi$ nor $\Gamma \vdash (\neg\phi)$. (On the other hand, for all assignment b and propositional formula ϕ, either $b(\phi) = 1$ or $b(\neg\phi) = 1$.) However,

1.7. **Lemma.** *If Γ is consistent, then there is no ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash (\neg\phi)$.*

Proof. If $\Gamma \vdash \phi$ and $\Gamma \vdash (\neg\phi)$ i.e., $\Gamma \vdash (\phi \to \bot)$ then (using mp) $\Gamma \vdash \bot$, i.e., Γ is not consistent. □

Notice that the opposite direction is also true: if Γ is not consistent, i.e., $\Gamma \vdash \bot$ then since for every formula ψ, $\emptyset \vdash (\bot \to \psi)$, we get that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$ for all ϕ.

1.8. **Lemma.** *Let ϕ be a propositional formula. If Γ is consistent, then either $\Gamma \cup \{\phi\}$ is consistent or $\Gamma \cup \{(\neg\phi)\}$ is consistent.*

Proof. If $\Gamma \cup \{\phi\}$ is not consistent, $\Gamma \cup \{\phi\} \vdash \bot$, i.e., by the Deduction Lemma, $\Gamma \vdash (\phi \to \bot)$, i.e.,

$$\Gamma \vdash (\neg\phi).$$

If $\Gamma \cup \{(\neg\phi)\}$ is also not consistent then we similarly get that

$$\Gamma \vdash (\neg(\neg\phi)),$$

hence using the axiom

$$((\neg(\neg\phi)) \to \phi)$$

and mp we get

$$\Gamma \vdash \phi.$$

Therefore we get $\Gamma \vdash (\neg\phi)$ and $\Gamma \vdash \phi$; by Lemma 1.7 this contradicts the assumption that Γ is consistent. □

1.9. **Lemma.** *Every set of formulas Γ decides the proposition $\bot$.*

Proof. We showed in PS2 that $\emptyset \vdash (\bot \to \bot)$, i.e., $\emptyset \vdash (\neg\bot)$. □

1.10. **Lemma.** *Let Γ be a set of propositional formulas, and let α, β be propositional formulas. If Γ decides α and decides β, then Γ decides $(\alpha \to \beta)$. Moreover, the decision corresponds to the truth table of $\to$, i.e.,*

(1) *If $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$, then $\Gamma \vdash (\alpha \to \beta)$.*
(2) *If $\Gamma \vdash (\neg\alpha)$ and $\Gamma \vdash \beta$, then $\Gamma \vdash (\alpha \to \beta)$.*
(3) *If $\Gamma \vdash (\neg\alpha)$ and $\Gamma \vdash (\neg\beta)$, then $\Gamma \vdash (\alpha \to \beta)$.*

(4) *If* $\Gamma \vdash \alpha$ *and* $\Gamma \vdash (\neg\beta)$, *then* $\Gamma \vdash (\neg(\alpha \to \beta))$.

Proof. (1) $\Gamma \vdash (\beta \to (\alpha \to \beta)$ (axiom), and $\Gamma \vdash \beta$ (given). Hence, by mp, $\Gamma \vdash (\alpha \to \beta)$.
(2) The same argument as in case (1).
(3) We saw in PS2 that $\Gamma \vdash ((\neg\alpha) \to (\alpha \to \beta))$. It is given that $\Gamma \vdash (\neg\alpha)$. Hence, by mp, $\Gamma \vdash (\alpha \to \beta)$.
(4) By the Deduction Lemma, it is enough to show that in this case $\Gamma \cup \{\alpha \to \beta)\} \vdash \bot$. The proof of this is left as an exercise.

□

A set of formulas Γ is *complete* if for every propositional formula ϕ Γ decides ϕ. Notice the difference between complete logic, like propositional logic is, i.e., the Completeness Theorem is satisfied, and complete set of formulas. Notice that if Γ is not consistent, then it is complete.

1.11. **Lemma.** *Let* Γ *be a set of propositional formulas. If* Γ *decides every propositional variable, then* Γ *is complete.*

Proof. Let Φ' be the set of formulas that are decided by Γ.

- $\bot \in \Phi'$ by Lemma 1.9.
- For every propositional variable p, $p \in \Phi'$, by the assumption of the lemma.
- If $\alpha, \beta \in \Phi'$ then $(\alpha \to \beta) \in \Phi'$ by Lemma 1.10.

Therefore Φ' is the set Φ of all propositional formulas. □

Recall that we assume that the set pf propositional variables is countable. Let $\{p_n : n \in \mathbb{N}\}$ be the set of propositional variables.

1.12. **Theorem.** *If* Γ *is consistent then there is* $\bar{\Gamma}$, *such that* $\Gamma \subseteq \bar{\Gamma}$, $\bar{\Gamma}$ *is consistent and complete.*

Proof. We define an increasing sequence of sets of formulas

$$\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$$

as follows.

$$\Gamma_0 = \Gamma.$$

If Γ_{n-1} is defined, define

$$\Gamma_n = \begin{cases} \Gamma_{n-1} \cup \{p_n\} & \text{if } \Gamma_{n-1} \cup \{p_n\} \text{ is consistent;} \\ \Gamma_{n-1} \cup \{(\neg p_n)\} & \text{otherwise.} \end{cases}$$

By induction, for all $n \in \mathbb{N}$, the set Γ_n is consistent: The set $\Gamma_0 = \Gamma$ is consistent by assumption. In the induction step, assume that Γ_{n-1} is consistent. If $\Gamma_{n-1} \cup \{p_n\}$ is consistent then, by definition, $\Gamma_n =$

$\Gamma_{n-1} \cup \{p_n\}$. If $\Gamma_{n-1} \cup \{p_n\}$ is not consistent, then by Lemma 1.8, $\Gamma_n = \Gamma_{n-1} \cup \{(\neg p_n)\}$ is consistent.

Set

$$\bar{\Gamma} = \cup_{n=1}^{\infty} \Gamma_n.$$

We claim that $\bar{\Gamma}$ is consistent. Assume not, that is, $\bar{\Gamma} \vdash \perp$, i.e., there exists a formal proof $\bar{\beta}$ of $\perp$ from $\bar{\Gamma}$. (Since $\bar{\beta}$ is finite), there is a finite number of elements of $\bar{\Gamma}$ in $\bar{\beta}$, each is in Γ_n for some $n \in \mathbb{N}$. Hence there is $N \in \mathbb{N}$ such that all of the elements of $\bar{\beta} \cap \bar{\Gamma}$ are in Γ_N, therefore $\Gamma_N \vdash \perp$ in contradiction with the fact that Γ_N is consistent.

We claim that $\bar{\Gamma}$ is complete. By Lemma 1.11, it is enough to see that every propositional variable p_n is decided by $\bar{\Gamma}$. Indeed, by definition of Γ_n, either $p_n \in \Gamma_n \subseteq \bar{\Gamma}$ or $(\neg p_n) \in \Gamma_n \subseteq \bar{\Gamma}$; in the first case $\Gamma \vdash p_n$, in the second case $\Gamma \vdash (\neg p_n)$. □

Proof of the Model Existence Theorem. Step 1. It is enough to show that if a complete set is consistent then it admits a model. Indeed, for a consistent Γ, there is $\bar{\Gamma} \supseteq \Gamma$ that is complete and consistent, by Theorem 1.12. If b is a model of $\bar{\Gamma}$, that is, $b(\phi) = 1$ for every $\phi \in \bar{\Gamma}$, then in particular, $b(\phi) = 1$ for every $\phi \in \Gamma \subseteq \bar{\Gamma}$, i.e., b is a model of Γ.

Step 2. Let Γ be a consistent and complete set. We define an assignment b:

$$b(p_n) = \begin{cases} 1 & \text{if } \Gamma \vdash p_n; \\ 0 & \text{if } \Gamma \vdash (\neg p_n). \end{cases}$$

Since we assume that Γ is complete and consistent, b is well defined.

Step 3. We show that b is a model of Γ by induction on the construction of a propositional formula. Set

$$\Phi' = \{\phi \mid b(\phi) = 1 \text{ iff } \Gamma \vdash \phi\}.$$

- For every $n \in \mathbb{N}$, the propositional variable p_n is in Φ', by definition of b (and Lemma 1.7).
- $b(\perp) = 0$ (as in every assignment), and there is no formal proof of $\perp$ from Γ, since Γ is consistent.
- If α, $\beta \in \Phi'$, then $(\alpha \to \beta) \in \Phi'$, by Lemma 1.10 (and Lemma 1.7).

Thus Φ' equals the set of all formulas, i.e., for every formula ϕ, $b(\phi) = 1$ iff $\Gamma \vdash \phi$. In particular, b is a model of Γ. □