18.510: INTRODUCTION TO MATHEMATICAL LOGIC AND SET THEORY, FALL 08

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1. Predicate Calculus, First-order Logic

Syntax. The *alphabet of a first-order language* contains the following symbols:

(1) variables: $v_1, v_2, \ldots$;
(2) logical symbols: $\bot$ (contradiction), $\rightarrow$ (if-then);
(3) equality symbol: $=$;
(4) quantifier: $\forall$ (for all);
(5)), ((parentheses);
(6)
- for every $n \geq 1$, a (possibly empty) set of n-ary relation symbols R_i^n;
- for every $n \geq 1$, a (possibly empty) set of n-ary function symbols f_j^n;
- a (possibly empty) set of constants.

In this course, we always assume that the alphabet is countable. We denote by $\mathcal{S}$ the set of symbols listed in item (6). We call it the *symbol set*. $\mathcal{S}$ may be empty or finite or countably infinite.

Given an alphabet with symbol set $\mathcal{S}$, we define terms and formulas.

We define the *terms* to be the strings over $\mathcal{S}$ which are obtained by finitely many applications of the following rules:

- Every variable is a term.
- Every constant is a term.
- If the strings $t_1, \ldots, t_n$ are terms and $f = f_j^n$ is an n-ary function symbol in $\mathcal{S}$ then $f(t_1, \ldots, t_n)$ is also a term.

We define the *formulas* to be the strings over $\mathcal{S}$ which are obtained by finitely many applications of the following rules:

(1) If t_1 and t_2 are terms then $t_1 = t_2$ is a formula.
(2) If $t_1, \ldots, t_n$ are terms and $R = R_i^n$ is an n-ary relation symbol then $R(t_1, \ldots, t_n)$ is a formula.
(3) $\bot$ is a formula.
(4) If ϕ and ψ are formulas then $(\phi \rightarrow \psi)$ is a formula.
(5) If ϕ is a formula and x is a variable then $(\forall x)\phi$ is a formula.

Formulas derived using item (1) and item (2) are called *atomic formulas* (because they are not formed combining other formulas). We denote the set of formulas by $\mathcal{L} = \mathcal{L}^{\mathcal{S}}$, and call it the *first-order language* associated with the alphabet $\mathcal{S}$.

1.1. *Remark.* (1) As before, for formulas α, β, we abbreviate $\neg\alpha$, $(\alpha \vee \beta)$, and $(\alpha \wedge \beta)$ for the corresponding formulas: $(\alpha \to \bot)$, $((\alpha \to \bot) \to \beta)$, and $\neg(\neg\alpha \vee \neg\beta)$.
(2) We also abbreviate $(\exists x)$ for $\neg(\forall x)\neg$.
(3) When the choice of the particular variables is unimportant, we will not specify the choice. Instead we will write, for example, $(\forall x)(\phi(x) \to \psi(x))$, where it is understood that x is some variable.

The terms and formulas are well defined: each has a unique decomposition into its constituents. See Theorem II.4.4 in the EFT textbook. Thus we can give inductive definitions and proofs on (the constructions of) terms or on formulas.

The *set of free variables* of a formula ϕ, denoted by $\mathrm{FV}(\phi)$, is defined as follows:

(1) $\mathrm{FV}(t_1 = t_2)$ is the set of all variables occurring in t_1, t_2.
(2) $\mathrm{FV}(R(t_1, \dots, t_n))$ is the set of all variables occurring in $t_1, \dots, t_n$.
(3) $\mathrm{FV}(\bot) = \emptyset$.
(4) $\mathrm{FV}((\phi \to \psi)) = \mathrm{FV}(\phi) \cup \mathrm{FV}(\psi)$.
(5) $\mathrm{FV}((\forall x)\phi) = \mathrm{FV}(\phi) \smallsetminus \{x\}$.

We set

$$L_n = L_n^{\mathcal{S}} := \{\phi \mid \phi \text{ is an } \mathcal{S}\text{-formula and } \mathrm{FV}(\phi) \subseteq \{v_1, \dots, v_n\}\}$$

Formulas without free variables are called *sentences*. By definition, if a formula is a sentence then either it has constants in place of variables, or its variables are bound, or both. For example, the formulas $c_1 = c_2$ and $(\forall x)R(x)$ (where R is a unary symbol relation) are sentences.

Semantics. Fix an alphabet with a symbol set $\mathcal{S}$. A *structure* $\mathcal{A}$ of $\mathcal{S}$ consists of the following:

- A nonempty set A: the *domain* or the *universe* of $\mathcal{A}$.
- For every n-ary relation symbol R_i^n in $\mathcal{S}$, the structure $\mathcal{A}$ associates an n-ary relation $(R_i^n)^{\mathcal{A}}$ on A, i.e., a subset $(R_i^n)^{\mathcal{A}} \subset A^n$.
- For every n-ary function symbol f_j^n in $\mathcal{S}$, the structure $\mathcal{A}$ associates an n-ary function $(f_j^n)^{\mathcal{A}}$ on A, i.e., a function $(f_j^n)^{\mathcal{A}} \colon A^n \to A$.
- For every constant c in $\mathcal{S}$, the structure $\mathcal{A}$ associates an element $c^{\mathcal{A}}$ of A.

1.2. *Example.* Consider the symbol sets

$$\mathcal{S}_{\text{ar}} := \{+, \cdot, 0, 1\} \text{ and } \mathcal{S}_{\text{ar}}^{<} := \{+, \cdot, 0, 1, <\}$$

where $+$ and $\cdot$ are binary function symbols, 0 and 1 are constants, and $<$ is a binary relation symbol.

We define the following structures.

(1) The $\mathcal{S}_{\text{ar}}$-structure

$$\mathcal{N} := (\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}})$$

where $+^{\mathbb{N}}$ and $\cdot^{\mathbb{N}}$ are the usual addition and multiplication on $\mathbb{N}$ and $0^{\mathbb{N}}$ and $1^{\mathbb{N}}$ are the numbers zero and one, respectively.

(2) The $\mathcal{S}_{\text{ar}}^{<}$-structure

$$\mathcal{N}^{<} := (\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}}, <^{\mathbb{N}})$$

where $<^{\mathbb{N}}$ is the usual ordering on $\mathbb{N}$. Similarly

(3)
$$\mathcal{R} := (\mathbb{R}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0^{\mathbb{R}}, 1^{\mathbb{R}})$$

(4)
$$\mathcal{R}^{<} := (\mathbb{R}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, 0^{\mathbb{R}}, 1^{\mathbb{R}}, <^{\mathbb{R}}).$$

(We will often omit the superscripts $\mathbb{N}$, $\mathbb{R}$ when discussing these structures.)

1.3. *Remark.* In formulas of $\mathcal{L}$ the variables refer to the elements of the domain of a structure. Given a structure, we often call elements of its domain A *first-order objects* while subsets of A are called *second-order objects*. Since $\mathcal{L}$ only has variables for first-order objects, we call $\mathcal{L}$ a first order language.

The satisfaction relation. For a formula ϕ, a structure $\mathcal{A}$, and an assignment $p\colon \text{FV}(\phi) \to A$ we say that $\mathcal{A}$ *is a model of* ϕ *with respect to the assignment* p, or ϕ is true *in* $\mathcal{A}$ *with respect to* p, denoted by $\mathcal{A} \models \phi(p)$, if the following holds.

(1) If $\phi = (t_1 = t_2)$, then

$$\mathcal{A} \models \phi(p) \text{ if } t_1{}^{\mathcal{A}}(a_1, \ldots, a_k) = t_2{}^{\mathcal{A}}(a_1, \ldots, a_k),$$

where $a_i = p(x_i),\ x_i \in \text{FV}(\phi)$.

(2) If $\phi = R_i^k(t_1, \ldots, t_k)$ then

$$\mathcal{A} \models \phi(p) \text{ if } (b_1, \ldots, b_k) \in R_i^k,$$

where $b_i = t_i^{\mathcal{A}}(a_1, \ldots, a_k),\ a_i = p(x_i),\ x_i \in \text{FV}(\phi)$.

(3) If $\phi = \bot$ then there is no structure $\mathcal{A}$ such that $\mathcal{A} \models \bot$.

(4) If $\phi = (\phi_1 \to \phi_2)$ and $p\colon \operatorname{FV}(\phi) \to A$, set p_1 to be the restriction

$$p_1 = p|_{\operatorname{FV}(\phi_1)}\colon \operatorname{FV}(\phi_1) \to A$$

and

$$p_2 = p|_{\operatorname{FV}(\phi_2)}\colon \operatorname{FV}(\phi_2) \to A.$$

By the induction assumption, the truth value of $\phi_1(p_1)$ and of $\phi_2(p_2)$ in $\mathcal{A}$ is already defined. We define the truth value of $\phi(p)$ in $\mathcal{A}$ according to the truth table of $\to$:

$\mathcal{A} \models \phi_1(p_1)$	$\mathcal{A} \models \phi_2(p_2)$	$\mathcal{A} \models (\phi_1 \to \phi_2)(p)$
0	0	1
0	1	1
1	0	0
1	1	1

(5) If $\phi = (\forall x)\psi$ and $p\colon \operatorname{FV}(\phi) \to A$, then p is not a full assignment for $\operatorname{FV}(\psi) = \operatorname{FV}(\phi) \cup \{x\}$. However, for every $a \in A$ we define an assignment $p_a\colon \operatorname{FV}(\psi) \to A$:

$$p_a(y) = \begin{cases} p(y) & \text{if } y \in \operatorname{FV}(\phi); \\ a & \text{if } y = x. \end{cases}$$

We say that

$$\mathcal{A} \models \phi(p) \text{ if for every } a \in A,\ \mathcal{A} \models \psi(p_a).$$

In particular, if ϕ is a sentence, i.e., $\operatorname{FV}(\phi) = \emptyset$ then the definition of $\mathcal{A} \models \phi$ requires no assignment.

1.4. *Example.* Consider $\mathcal{R}^< = \{\mathbb{R}, +, \cdot, 0, 1, <\}$. Set

$$\phi(x_1, x_2) = (\exists x_1)(\forall x_2)(x_1 \cdot x_1 + x_2 \cdot x_2 > 1)$$

($\operatorname{FV}(\phi) = \emptyset$). We find the truth value of ϕ in $\mathcal{A}$ in a recursive process:

(1) For $\phi_1 = (x_1 \cdot x_1 + x_2 \cdot x_2 > 1)$, $\operatorname{FV}(\phi_1) = \{x_1, x_2\}$, an assignment $p\colon \{x_1, x_2\} \to \mathbb{R}$ is a choice of a point $(a_1, a_2) \in \mathbb{R}^2$. $\mathcal{R}^< \models \phi_1(p)$ if (and only if) this point is out of the unit circle.

(2) For $\phi_2 = (\forall x_2)(x_1 \cdot x_1 + x_2 \cdot x_2 > 1)$, $\operatorname{FV}(\phi_2) = \{x_1\}$, an assignment $p\colon \{x_1\} \to \mathbb{R}$ is a choice of a point $a_1 \in \mathbb{R}$. Then $\mathcal{R}^< \models \phi_2(p)$ iff for all $a_2 \in \mathbb{R}$, (a_1, a_2) is out of the unit circle in $\mathbb{R}^2$. Hence $\mathcal{R}^< \models \phi_2(p)$ iff p sends x_1 to a point out of the interval $[-1, 1]$ in $\mathbb{R}$, i.e., to a point from the right of 1 and the left of -1, (not including 1 and -1).

(3) For $\phi_3 = \neg(\forall x_2)(x_1 \cdot x_1 + x_2 \cdot x_2 > 1) = (\phi_2 \to \bot)$ and an assignment $p\colon \{x_1\} \to \mathbb{R}$, the truth value of $\phi_3(p)$ is by the truth table of $\bot$ and $\to$:

$\mathcal{R}^< \models \phi_2(p_1)$	$\mathcal{R}^< \models \bot$	$\mathcal{R}^< \models \phi_3(p)$
0	0	1
1	0	0

Therefore, $\mathcal{R}^< \models \phi_3(p)$ iff $\mathcal{R}^<$ is not a model of $\phi_2(p)$, i.e., iff p sends x_1 to a point in the interval $[-1, 1]$ (including the endpoints).

(4) For $\phi_4 = (\forall x_1)\neg(\forall x_2)(x_1 \cdot x_1 + x_2 \cdot x_2 > 1) = (\forall x_1)(\phi_2 \to \bot) = (\forall x_1)\phi_3$. Then $\mathcal{R}^< \models \phi$ if for every $a \in \mathbb{R}$, $\mathcal{R}^< \models \phi_3(p_a)$ where $p_a(x_1) = a$. Therefore $\mathcal{R}^<$ is not a model of ϕ_4 (since, e.g., for $a = 2$, $(\forall x_2)(2 \cdot 2 + x_2 \cdot x_2 > 1)$).

(5) For $\phi = \neg\phi_4 = \neg(\forall x_1)\neg(\forall x_2)(x_1 \cdot x_1 + x_2 \cdot x_2 > 1)$, $\mathcal{R}^< \models \phi$ iff $\mathcal{R}^< \not\models \phi_4$. Therefore $\mathcal{R}^< \models \phi$.

For a set of sentences Γ we say that a structure $\mathcal{A}$ is a *model* of Γ, and write $\mathcal{A} \models \Gamma$ if $\mathcal{A} \models \psi$ for every $\psi \in \Gamma$.

1.5. *Example.* A group is a structure $(G, \circ^G, e^G)$ in the symbol set $\mathcal{S}_{\text{gr}} := \{\circ, e\}$, where $\circ$ is a binary function, and e is a constant, that is a model of the set of sentences Φ_{gr}:

(1) $(\forall x)(\forall y)(\forall z)(x \circ y) \circ z = x \circ (y \circ z)$
(2) $(\forall x)x \circ e = x$
(3) $(\forall x)(\exists y)x \circ y = e$

1.6. *Example.* An equivalence structure is a structure (A, R^A) in the symbol set $\mathcal{S}_{\text{eq}} := \{R\}$, where R is a binary relation, that is a model of the set of sentences:

(1) $(\forall x)xRx$
(2) $(\forall x)(\forall y)(xRy \to yRx)$
(3) $(\forall x)(\forall y)(\forall z)((xRy \wedge yRz) \to xRz)$

1.7. *Example.* PA: Consider the structure $\mathcal{N}^s := (\mathbb{N}, +, \cdot, 0, s)$, where we interpret s as the successor function $s(n) = n + 1$ for $n \in \mathbb{N}$. The structure $\mathcal{N}^s$ satisfies the *Peano axiom system* Φ_{PA}:

(1) $(\forall x_1)(\forall x_2)((s(x_1) = s(x_2)) \to (x_1 = x_2))$ (i.e., s is one-to-one).
(2)
- $(\forall x)(\neg(x = 0) \to (\exists y)(s(y) = x))$ (i.e., every element in $\mathbb{N}$ that is not 0 is a successor of another element).
- $\neg(\exists y)(s(y) = 0)$ (i.e., there is no element of $\mathbb{N}$ such that 0 is its successor).

(3)
- $(\forall y)(0 + y = y)$
- $(\forall x)(\forall y)(s(x) + y = s(x + y))$

(4)
- $(\forall y)(0 \cdot y = 0)$
- $(\forall x)(\forall y)(s(x) \cdot y = x \cdot y + y)$

(5) Induction principle: for all $x, y_1, \ldots, y_n$ and ϕ with $\mathrm{FV}(\phi) = \{x, y_1, \ldots, y_n\}$,

$(\forall \bar{y})([\phi(0, \bar{y}) \wedge (\forall z)(\phi(z, \bar{y}) \rightarrow \phi(s(z), \bar{y}))] \rightarrow (\forall x)\phi(x, \bar{y}))$,

where $\bar{y} = \{y_1, \ldots, y_n\}$ and $(\forall \bar{y}) = (\forall y_1) \ldots (\forall y_n)$.

The consequence relation.

- For a set of sentences Γ and a sentence ϕ we say that ϕ is a *consequence* of Γ, and write $\Gamma \models \phi$ if every structure that is a model of Γ is a model of ϕ. (If $\Gamma = \{\psi\}$ then we write $\psi \models \phi$ instead of $\{\psi\} \models \phi$.)
- For a set of sentences Γ and a formula ϕ with $\mathrm{FV}(\phi) = (x_1, \ldots, x_n)$ we say that ϕ is a *consequence* of Γ, and write $\Gamma \models \phi$ if $\Gamma \models (\forall x_1) \ldots (\forall x_n)\phi$.
- A formula ϕ is valid, written $\models \phi$, if $\emptyset \models \phi$.
- Formulas ϕ_1, ϕ_2 are *logically equivalent* if $\models (\phi_1 \rightarrow \phi_2)$ and $\models (\phi_2 \rightarrow \phi_1)$.

To show that a formula ϕ is not a consequence of a set of sentences Γ, it is sufficient to give a structure and an assignment which satisfy every sentence of Γ but fails to satisfy ϕ.

1.8. *Example.* Let $\Gamma = \Phi_{\mathrm{gr}}$ (see Example 1.5 above). Then if $\mathcal{G} = (G, \circ, e)$ is not Abelian, $\mathcal{G} \not\models (\forall x)(\forall y)(x \circ y = y \circ x)$ and $\mathcal{G} \models \Phi_{\mathrm{gr}}$, hence $\Phi_{\mathrm{gr}} \not\models (\forall x)(\forall y)(x \circ y = y \circ x)$. Analogously, one can use an Abelian group to show that $\Phi_{\mathrm{gr}} \not\models \neg(\forall x)(\forall y)(x \circ y = y \circ x)$

The above example shows that in first-order logic (as in propositional logic), it might be that $\Gamma \not\models \phi$ and $\Gamma \not\models \neg\phi$.