18.510: INTRODUCTION TO MATHEMATICAL LOGIC AND SET THEORY, FALL 08

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1. Predicate Calculus, First-order Logic

Semantics.

Homomorphism and Isomorphism. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{S}$-structures. A map $h\colon A \to B$ is called an *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if

- For n-ary $R \in \mathcal{S}$ and $a_1, \ldots, a_n \in A$,
$$R^{\mathcal{A}}(a_1, \ldots, a_n) \text{ iff } R^{\mathcal{B}}(h(a_1), \ldots, h(a_n)).$$
- For n-ary $f \in \mathcal{S}$ and $a_1, \ldots, a_n \in a$,
$$h(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(h(a_1), \ldots, h(a_n)).$$
- For $c \in \mathcal{S}$,
$$h(c^{\mathcal{A}}) = c^{\mathcal{B}}.$$

If h is also one-to-one and onto we call it an *isomorphism*, and write $h\colon A \cong B$. $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic* if there is an isomorphism $h\colon A \cong B$.

For example, the $\mathcal{S}_{\mathrm{gr}}$-structure $(\mathbb{N}, +, 0)$ is isomorphic to the $\mathcal{S}_{\mathrm{gr}}$-structure $(G, +^G, 0)$ consisting of the even natural numbers with ordinary addition as $+^G$. The map $\pi\colon \mathbb{N} \to G$ mapping $\pi(n) = 2n$ is an isomorphism of $(\mathbb{N}, +, 0)$ onto $(G, +^G, 0)$.

1.1. **Lemma** (Isomorphism Lemma)**.** *If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic $\mathcal{S}$-structures then for all $\mathcal{S}$-sentences ϕ*
$$\mathcal{A} \models \phi \textit{ iff } \mathcal{B} \models \phi.$$

You will prove this lemma in PS4.

1.2. *Remark.* However, two $\mathcal{S}$-structures might satisfy the same $\mathcal{S}$-sentences but not be isomorphic. We will see an example later in the course.

Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{S}$-structures. $\mathcal{A}$ is called a *substructure* of $\mathcal{B}$, written $\mathcal{A} \subset \mathcal{B}$, if $A \subset B$ and the inclusion map $i\colon A \hookrightarrow B$ $(i(a) = a)$ is an homomorphism.

For example, $(\mathbb{Z}, +, 0)$ is a substructure of $(\mathbb{Q}, +, 0)$, and $(\mathbb{N}, +, 0)$ is a substructure of $(\mathbb{Z}, +, 0)$ (although $(\mathbb{N}, +, 0)$ is not a subgroup of $(\mathbb{Z}, +, 0)$). Notice that the sentence

$$(\forall x)(\exists y) y + y = x$$

holds in $(\mathbb{Q}, +, 0)$, but not in $(\mathbb{N}, +, 0)$. So sentences might no longer hold when passing to substructures.

1.3. **Lemma.** *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{S}$ structure with $\mathcal{A} \subset \mathcal{B}$ and let $p\colon \{v_n\} \to A$ be an assignment in $\mathcal{A}$. Then for every quantifier-free $\mathcal{S}$-formula ϕ:*

$$\mathcal{A} \models \phi(p) \text{ iff } \mathcal{B} \models \phi(p).$$

You will prove this lemma in PS4.