18.510: INTRODUCTION TO MATHEMATICAL LOGIC AND SET THEORY, FALL 08

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1. Predicate Calculus, First-order Logic

Recall: when discussing first-order logic we always assume that we set a symbol set $\mathcal{S}$.

Axioms. We declare the following valid sentences to be *axioms*.

(1) All the axioms from propositional logic, e.g. $(\neg\neg\phi \to \phi)$, which now include sentences that did not exist in propositional logic, e.g., $(\neg\neg(\forall x)R(x,y) \to (\forall x)R(x,y))$.

(2) Equality axioms:
   (a) $(\forall x)(x = x)$.
   (b) $(\forall x)(\forall y)(x = y \to y = x)$.
   (c) $(\forall x)(\forall y)(\forall z)((x = y) \wedge (y = z) \to (x = z))$.
   (d) $(\forall x))(\forall y)((x = y) \to (\phi \to \phi'))$ where ϕ' is obtained from ϕ by substituting x for y.

(3) $((\forall x)\phi \to \phi[\frac{a}{x}])$ where $\phi[\frac{a}{x}]$ is the formula obtained by replacing all the instances of x in ϕ by a.

1.1. If t is a term and ϕ is a formula possibly containing the variable x, then $\phi[\frac{t}{x}]$ is the result of replacing all instances of x by t in ϕ. The convention is that this is done only when x is free. This replacement results in a formula that logically follows the original one provided that no free variable of t becomes bound in this process. If some free variable of t becomes bound, then to substitute t for x it is first necessary to change the names of bound variables of ϕ to something other than the variables of t. Forgetting this condition is a notorious cause of errors.

To see why this condition is necessary, consider the formula ϕ given by $(\exists z)z+z = x$. In $\mathcal{N}$ the formula ϕ says that x is even. If we replace x by y in ϕ we obtain the formula $(\exists z)z + z = y$ which says that y is even. But if we replace the variable x by z, we obtain the formula $(\exists z)z+z = z$, which no longer says that z is even; in fact, the resulting formula is a sentence valid in $\mathcal{N}$ (since $0 + 0 = 0$).

First-order Logic: formal proofs. We say that a sequence $\bar{\beta}$ of finitely many sentences, $\bar{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, is a *formal proof* of a sentence ϕ from a set Γ of sentences if $\beta_n = \phi$ and for all i, either

(1) β_i is an axiom;
(2) $\beta_i \in \Gamma$;
(3) (mp) $\beta_i = \gamma$, and there are $j, k < i$ such that $\beta_j = (\alpha \to \gamma)$ and $\beta_k = \alpha$.
(4) $\beta_i = (\neg(\forall x)\psi \to \neg\psi(\frac{c}{x}))$ where c is a constant symbol that does not occur in Γ, nor in ψ, nor in ϕ, nor in any β_j for $j < i$.

A sentence ϕ is *formally provable* or *derivable* from a set Γ of sentences, written $\Gamma \vdash \phi$, if there exists a formal proof of ϕ from Γ.

1.2. *Remark.* If the sentences in $\Gamma \cup \{\phi\}$ are in a language $\mathcal{L}$, and $\Gamma \vdash \phi$, there might be sentences in the proof that are not in $\mathcal{L}$: sometimes we will have to add symbols to the language for new constants.

1.3. *Remark.* If $\Gamma \vdash \phi$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash \phi$, but maybe with a slightly different proof: if a proof $(\beta_1, \ldots, \beta_m)$ included a step $\beta_i = (\neg(\forall x)\psi \to \neg\psi(\frac{c}{x}))$with a constant symbol c that does not occur in Γ but does occur in Γ' we have to replace c with c' that is not in Γ' (nor in ψ nor in β_j for $j < i$).

1.4. **Lemma.** *Generalization on constants. Assume that* $\Gamma \vdash \psi[\frac{c}{x}]$, *where the constant symbol* c *does not occur in* Γ *nor in* ψ. *Then* $\Gamma \vdash (\forall x)\psi(x)$.

Proof.

$(\neg(\forall x)\psi(x) \to \neg\psi[\frac{c}{x}])$	type (4)
$((\exists x)\neg\psi(x) \to (\psi[\frac{c}{x}] \to \bot))$	explain
$(((\exists x)\neg\psi(x) \to (\psi[\frac{c}{x}] \to \bot)) \to (((\exists x)\neg\psi(x) \to \psi[\frac{c}{x}]) \to \neg(\exists x)\neg\psi(x)))$	axiom
$(((\exists x)\neg\psi(x) \to \psi[\frac{c}{x}]) \to ((\exists x)\neg\psi(x) \to \bot))$	mp
$(\psi[\frac{c}{x}] \to ((\exists x)\neg\psi(x) \to \psi[\frac{c}{x}]))$	axiom
$\psi[\frac{c}{x}]$	$(\Gamma \vdash \psi[\frac{c}{x}])$
$((\exists x)\neg\psi(x) \to \psi[\frac{c}{x}])$	mp
$((\exists x)\neg\psi(x) \to \bot)$	mp
$(\forall x)\psi(x)$	explain

□

1.5. **Theorem.** *Let* Γ *be a set of sentences in* $\mathcal{L}_{\mathcal{S}}$ *and* ϕ *a sentence in* $\mathcal{L}_{\mathcal{S}}$. *If* $\Gamma \vdash \phi$ *then* $\Gamma \models \phi$.

Assume $\Gamma \vdash \phi$ with $\beta = (\beta_1, \ldots, \beta_n)$. We will show that $\Gamma \models \phi$.

We can assume that if a constant c occurs in the proof β but does not occur in Γ nor in ϕ it is new, i.e., not in the symbol set $\mathcal{S}$. If not, replace c (wherever it occurs) with c' not in $\mathcal{S}$. Now, every step in the proof of type (4) uses a constant that is new to $\mathcal{S}$. Assume that steps of type (4) are $\beta_{i_1}, \ldots, \beta_{i_k}$ with corresponding constants $c_1, \ldots, c_k$. Set $\mathcal{S}' = \mathcal{S} \cup \{c_1, \ldots, c_k\}$.

1.6. **Claim.** *Let an $\mathcal{S}$-structure $\mathcal{A}$ be a model of Γ. Then there are elements $a_1, \ldots, a_k \in A$ such that the $\mathcal{S}'$-structure $\mathcal{A}'$ defined as $\mathcal{A} + (c_i^{\mathcal{A}'} = a_i)_{1 \le i \le k}$ is a model of β_j for all $1 \le j \le n$.*

Proof. By induction on j. Assume that there is an $\mathcal{S} \cup \{c_1, \ldots, c_{\ell-1}\}$-structure $\mathcal{A}'_{j-1}$ in which there is an interpretation to the constants $c_1, \ldots, c_{\ell-1}$ such that $\mathcal{A}'_{j-1} \models \beta_i$, for all $1 \le i \le (j-1)$, and $i_{(\ell-1)} \le (j-1) < i_\ell$. Then

(1) if $\beta_j \in \Gamma$ then $\mathcal{A} \models \beta_j$ (since $\mathcal{A} \models \Gamma$) hence also $\mathcal{A}'_{j-1} \models \beta_j$ (the new constants have nothing to do with β_j).
(2) if β_j is an axiom it is true in every structure (of $\mathcal{S} \subseteq \mathcal{S}'$), and does not contain any of the new constants, hence it is true in $\mathcal{A}'_{j-1}$.
(3) if β_j was obtained by mp, we use the fact that mp preserves $\models$, as proved in pset.
(4) If $\beta_j = \beta_{i_\ell} = (\neg(\forall x)\psi \to \neg\psi(\frac{c_\ell}{x}))$ then either
   - $\mathcal{A}'_{j-1} \models (\forall x)\psi$, and then we can interpret c_ℓ as we wish; for every interpretation a_ℓ of c_ℓ, it is true that $\mathcal{A}'_j = \mathcal{A}'_{j-1} + ({c_\ell}^{\mathcal{A}'_j} = a_\ell) \models (\neg(\forall x)\psi \to \neg\psi(\frac{c_\ell}{x}))$.
   - Or $\mathcal{A}'_{j-1} \not\models (\forall x)\psi$, i.e., *not* for every assignment $p\colon \{x\} \to A$ we have $\mathcal{A}'_{j-1} \models \psi(p)$. Therefore there is an assignment p such that $p(x) = a$ and $\mathcal{A}'_{j-1} \models \neg\psi[\frac{a}{x}]$. We set $\mathcal{A}'_j = \mathcal{A}'_{j-1} + ({c_\ell}^{\mathcal{A}'_j} = a)$. Then, $\mathcal{A}'_j \models \neg\psi(\frac{c_\ell}{x})$ hence $\mathcal{A}'_j \models (\neg(\forall x)\psi \to \neg\psi(\frac{c_\ell}{x}))$.

In particular, $\mathcal{A}_n \models \beta_n = \phi$. □

Now, notice that since none of the new constants $c_1, \ldots, c_k$ occur in ϕ their interpretation does not matter to determine whether an $\mathcal{S}$-structure is a model of ϕ. We deduce that $\mathcal{A} \models \phi$. This completes the proof of the theorem.

Embedding of propositional logic in first-order logic. Set a symbol-set $\mathcal{S}$ and let $\mathcal{L}_\mathcal{S}$ be the first-order language associated with $\mathcal{S}$.

1.7. **Question.** *How can we see a sentence in $\mathcal{L}_\mathcal{S}$ as a proposition in propositional logic?*

In propositional logic there are only the logic symbols $\bot$, $\to$ (and propositional variables). So we will think of sentences involving other symbols as propositional variables. For example

$$((\forall x)R(x,x) \to \neg(\forall y)S(y))$$

is read as $(p \to \neg q)$, where p stands for $(\forall x)R(x,x)$ and q stands for $(\forall y)S(y)$. When the same expression occurs again we read it the same way. For example, $((\forall x)R(x,x) \to (\forall x)R(x,x))$ is read $(p \to p)$.

From the point of view of propositional logic, every first-order atomic sentence is read as a propositional variable, and every sentence that starts with $(\forall x)$ is read as a propositional variable.

Notice that every axiom and inference-rule of propositional logic hold in first-order logic. In particular, if Γ is a set of first-order sentences and is consistent in first order logic, then it is consistent (when read as a set of propositions) in propositional logic.

1.8. **Question.** *How do we think about a first-order structure $\mathcal{A}$ as an assignment in propositional logic?*

For every "propositional variable" (see above) ϕ we define

$$V_{\mathcal{A}}(\phi) = \begin{cases} 1 & \text{if } \mathcal{A} \models \phi; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that every sentence ϕ has a truth-value according to $\mathcal{A}$ and according to $V_{\mathcal{A}}$, and they are the same (by definition, and since both are determined by the truth-tables of $\to$ and $\perp$).

A first-order sentence is called a *tautology* if its translation to propositional logic is a tautology. The set of tautologies is a proper subset of the set of valid sentences. Notice that in every structure $\mathcal{A}$, a tautology ψ is assigned truth value 1 in $\mathcal{A}$ since $V_{\mathcal{A}}(\psi) = 1$.

1.9. *Remark.* We will show later that if the symbol-set $\mathcal{S}$ is countable then the set of first-order sentences over $\mathcal{S}$ is countable. In particular, the set of propositional variables in the translation to propositional logic is countable.

1.10. **Lemma** (First-order Deduction Lemma (in proofs))**.** *Let Γ be a set of sentences, and let ϕ, ψ be sentences. If $\Gamma \cup \{\phi\} \vdash \psi$ then $\Gamma \vdash (\phi \to \psi)$.*

Proof. We assume that there exists a proof $\alpha = (\alpha_1, \ldots, \alpha_m)$ of ψ from $\Gamma \cup \{\phi\}$. Denote by

$$\Delta = \{\alpha_{i_1}, \ldots, \alpha_{i_n}\}$$

the set of lines α_i that are justified as axioms or as type (4) .

1.11. **Claim.** *$\Delta \cup \Gamma \cup \{\phi\} \vdash \psi$ when embedded in propositional logic.*

Proof. The proof α is justifies as a proof of ψ from $\Delta \cup \Gamma \cup \{\phi\}$ in propositional logic:

(1) If α_i is justified by mp then it is justified by mp in propositional logic.
(2) If α_i is justified as an assumption (i.e., an element of $\Gamma \cup \{\phi\}$) it is justified as an assumption in propositional logic.
(3) If α_i is justified as an axiom or as type (4), then it is justified as an element of Δ in propositional logic.

□

Now, by the Deduction Lemma in propositional logic, $\Delta \cup \Gamma \vdash (\phi \to \psi)$. Let $\beta = (\beta_1, \ldots, \beta_k)$ be a proof of $(\phi \to \psi)$ from $\Delta \cup \Gamma$ in propositional logic. Then $(\alpha_{i_1}, \ldots, \alpha_{i_n}, \beta_1, \ldots, \beta_k)$ is a proof of $(\phi \to \psi)$ from Γ in first-order logic. (Every α_{i_j} is justified as an axiom or as type (4).)

□

1.12. **Lemma.** *Contraposition Lemma.* $\Gamma \cup \{\alpha\} \vdash \neg\beta \Leftrightarrow \Gamma \cup \{\beta\} \vdash \neg\alpha$.

Proof: Exercise (pset).