18.510: INTRODUCTION TO MATHEMATICAL LOGIC AND SET THEORY, FALL 08

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1. Predicate Calculus, First-order Logic

Proof of the general case of the Model Existence Theorem. Let Γ be a consistent set of sentences. In order to find a model $\mathcal{A}$ of Γ, we have at our disposal only the syntactical information given by the consistency of Γ. A first idea is to take as domain A the set

$$F_0$$

of all terms t such that $\mathrm{FV}(t) = \emptyset$, and to interpret, for instance, a unary function symbol f by

$$f^{\mathcal{A}}(t) := f(t),$$

and a unary relation symbol R by $R^{\mathcal{A}} := \{t \in A \mid \Gamma \vdash R(t)\}$. Here a first difficulty arises concerning the equality symbol: If t_1, t_2 are different terms then $t_1 \neq t_1$, hence ${t_1}^{\mathcal{A}} \neq {t_2}^{\mathcal{A}}$. If Γ is such that $\Gamma \vdash t_1 = t_2$, then $\mathcal{A}$ is not a model of Γ. We overcome this difficulty by defining an equivalence relation on terms and then using the equivalences classes rather than the individual terms as elements of the domain of $\mathcal{A}$.

Fix a set of sentences Γ. For terms s, t in F_0, we write $s \sim t$ if $\Gamma \vdash (s = t)$.

1.1. **Lemma.** (1) $\sim$ *is an equivalence relation on* F_0*, i.e.,*
   (a) $s \sim s$ *for all* $s \in F_0$.
   (b) $s \sim t \Rightarrow t \sim s$ *for all* $s, t \in F_0$
   (c) $s \sim t$, $t \sim u \Rightarrow s \sim u$ *for all* $s, t, u \in F_0$.
 (2) $\sim$ *is a normal equivalence relation, i.e., If* $t_1 \sim s_1, \dots, t_k \sim s_k$, *and* F_l^k *is a function symbol, then* $F_l^k(t_1, \dots, t_k) \sim F_l^k(s_1, \dots, s_k)$.

You prove part (1) of this lemma in PS 5.

Part (2) of Lemma 1.1 follows from the following lemma (with $\phi = F_l^k(t_1, \dots, t_k) = F_l^k(x_1, \dots, x_k)$).

1.2. **Lemma.** *If* $t_1 \sim s_1, \dots, t_k \sim s_k$, *and* ϕ *is a formula with* $\mathrm{FV}(\phi) = \{x_1, \dots, x_k\}$, *then if* $\Gamma \vdash \phi(t_1, \dots, t_k)$ *then* $\Gamma \vdash \phi(s_1, \dots, s_k)$

You prove Lemma 1.2 in PS 5.

We write $\bar{t} = \frac{t}{\sim}$, the equivalence class of $t \in F_0$ ($\bar{t} = \bar{s} \Leftrightarrow t \sim s$). We write A_Γ for the quotient algebra $\{\bar{s} \mid s \in F_0\}$. We saw that $s \to \bar{s}$ is a homomorphism (onto) $F_0 \to A_\Gamma$.

Proof of the general case of the Model Existence Theorem. Let $\Gamma \subseteq \Gamma_1$ be a set of sentences over $\mathcal{S}+$ that is consistent and contains witnesses. Let $\bar{\Gamma}$ be the set of all sentences that can be proven from Γ_1 (in particular, $\bar{\Gamma}$ contains all the axioms over $\mathcal{S}+$). Notice that $\bar{\Gamma}$ is still consistent (If $\bar{\Gamma} \vdash \perp$ then $\Gamma_1 \vdash \perp$ with the same proof.)

We define a model $\mathcal{A}$. The domain is the quotient algebra $A_{\bar{\Gamma}}$. We interpret the function symbols in the symbol set according to $A_{\bar{\Gamma}}$, For every relation symbol R_l^k we interpret $R_l^{k\mathcal{A}} \subseteq A_{\bar{\Gamma}}^k$ as follows:

$$(\bar{t_1}, \ldots, \bar{t_k}) \in R_l^{k\mathcal{A}} \text{ iff } \bar{\Gamma} \vdash R_l^k(t_1, \ldots, t_k).$$

By Lemma 1.2, $R_l^{k\mathcal{A}}$ is well defined.

1.3. **Claim.** *For every formula ϕ over $\mathcal{S}+$ with* $\mathrm{FV}(\phi) = \{x_1, \ldots, x_k\}$, *and every* $(t_1, \ldots, t_k) \in F_o$,

$$\mathcal{A} \models \phi(\bar{t_1}, \ldots, \bar{t_k}) \Leftrightarrow \bar{\Gamma} \vdash \phi[\frac{t_1}{x_1}, \ldots, \frac{t_k}{x_k}].$$

Proof. We prove the claim by induction on ϕ. First, let ϕ be an atomic formula, i.e.,

(1) $\phi = R_l^m(s_1, \ldots, s_m)$, or
(2) $\phi = (s = t)$.

In PS 5 you prove that in both these cases, the claim holds.

The induction step (and the proof for $\phi = \perp$) is the same as in the proof of the special case. (Check!) □

In particular, for every $\phi \in \Gamma$, $\mathcal{A} \models \phi$. Hence the restriction of $\mathcal{A}$ to the symbol set $\mathcal{S}$ of Γ is a model of Γ. □

1.4. *Remark.* We showed in the previous lecture that in first order logic, the Model Existence Theorem implies the Completeness Theorem, as it did in Propositional logic (with the same proof). Similarly, the statement of the Compactness Theorem and its proof, given the Model Existence Theorem or the Completeness Theorem, in first order logic, are the same as in propositional logic.