18.510: INTRODUCTION TO MATHEMATICAL LOGIC AND SET THEORY, FALL 08

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1. Predicate Calculus, First-order Logic

The Completeness Theorem.

1.1. **Theorem** (Model Existence Theorem)**.** *If* Γ *is consistent then it admits a model.*

1.2. **Theorem** (Completeness Theorem)**.** *If* $\Gamma \models \phi$ *then* $\Gamma \vdash \phi$*.*

Proof. Given $\Gamma \models \phi$. We want to show $\Gamma \cup \{\neg\phi\} \vdash \bot$, which happens iff $\Gamma \vdash \phi$ (check). Suppose by contradiction that $\Gamma \cup \{\neg\phi\}$ is consistent. Then by the Model Existence theorem, there is a structure $\mathcal{A}$ such that $\mathcal{A} \models \Gamma$ and $\mathcal{A} \models \neg\phi$ in contradiction with $\Gamma \models \phi$. □

1.3. *Remark.* The Completeness Theorem is about truth in the set of all structures, not about truth in a given structure. For example, it does not answer whether $\mathcal{N} \models \phi$.

Proof of the Model Existence Theorem. A set of sentences Γ *contains witnesses* if for every formula ϕ with $\mathrm{FV}(\phi) = \{x\}$, there is a constant $c \in \mathcal{S}$ such that $(\neg(\forall x)\phi \to \neg\phi[\frac{c}{x}]) \in \Gamma$.

The sentence $\psi = (\neg(\forall x)\phi \to \neg\phi(\frac{c}{x}))$ is giving a *witness* to the unhappening of the universal sentence $(\forall x)\phi$.

1.4. **Lemma.** *If* Γ *is consistent,* $\mathrm{FV}(\phi) = \{x\}$*, and* c *is a new constant that does not occur in* Γ *nor in* ϕ *then* $\Gamma \cup \{(\neg(\forall x)\phi \to \neg\phi(\frac{c}{x}))\}$ *is consistent.*

Proof. Suppose for contradiction that $\Gamma \cup \{\psi\} \vdash \bot$ with proof $\beta = (\beta_1, \ldots, \beta_n)$. If the proof does not use ψ then $\Gamma \vdash \bot$ by β, in contradiction with the fact that Γ is consistent.

Assume that ψ is one of the steps in β, justified as an assumption (an element of $\Gamma \cup \{\psi\}$). We can assume that $\beta_1 = \psi$. Then we can write the same proof from Γ, but this time justifying the first line β_1 as a step of type (4). (We assumed that c does not occur in Γ, and it definitely does not occur in $\bot$.) The rest of the lines in the proof β are justified as before and we get a contradiction again.

□

Notation:

Given $\mathcal{S}$ we write $\mathcal{S}+ = \mathcal{S} \cup \{c_1, c_2, \ldots\}$, where c_i, $i \in \mathbb{N}$ are new constant symbols.

1.5. *Remark.* If a symbol set is countable then the set of sentences in the first-order language is countable. We will prove this later.

1.6. **Lemma.** *If* Γ *is a consistent set of sentences over* $\mathcal{S}$ *then there is* $\Gamma \subseteq \bar{\Gamma}$*, a set of sentences over* $\mathcal{S}+$ *that is consistent and contains witnesses.*

Proof. By Remark 1.5, we can write all the sentences over $\mathcal{S}+$ as a sequence $\phi_1, \phi_2, \ldots$ In each ϕ_i there are at most finitely many constant symbols: there exists $n(i)$ such that for every constant symbol c_j that occurs at ϕ_i, $j < n(i)$. Let

$$N(i) = 1 + \max\{n(1), \ldots, n(i)\}.$$

Then $c_{N(i)}$ does not occur in $\Gamma \cup \{\phi_1, \ldots, \phi_n\}$.

We define sentences ψ_i by induction on i. Assume that we defined $\psi_1, \ldots, \psi_{n-1}$ and that there is $m(i) \geq N(i)$ such that if c_j occurs in (at least) one of the sentences $\psi_1, \ldots, \psi_{i-1}$ then $j < m(i)$. Then if $\phi_i = \neg(\forall x)\theta$, we set

$$\psi_i = (\neg(\forall x)\theta \to \neg\theta[\frac{c_{m_i}}{x}]),$$

i.e., ψ_i is a witness for ϕ_i. (If ϕ_i is not of that form, we do not need a witness, set $\psi_i = (\forall x)(x = x)$.) The constant $c_{m(i)}$ in ψ_i is chosen such that it does not appear in $\mathcal{S}$ nor in $\psi_1, \ldots, \psi_{i-1}$.

1.7. **Claim.** $\Gamma \cup \{\psi_1, \ldots, \psi_i\}$ *is consistent.*

Proof of the Claim. The proof is by induction on i. In the induction step, we assume that $\Gamma_i = \Gamma \cup \{\psi_1, \ldots, \psi_{i-1}\}$ is consistent, hence by Lemma 1.4 (and the definition of ψ_i), $\Gamma \cup \{\psi_1, \ldots, \psi_i\}$ is also consistent. □

We define

$$\bar{\Gamma} = \Gamma \cup \{\psi_1, \psi_2, \ldots\}.$$

We claim that $\bar{\Gamma}$ is consistent. Suppose not, i.e., $\bar{\Gamma} \vdash \bot$. The proof uses a finite number of elements of $\bar{\Gamma}$, all in $\Gamma \cup \{\psi_1, \ldots, \psi_m\}$ (for some $m \in \mathbb{N}$). Hence the same proof of $\bot$ is justified as a proof from $\Gamma \cup \{\psi_1, \ldots, \psi_m\}$, in contradiction with the claim above.

We claim that $\bar{\Gamma}$ contains witnesses. Indeed, if θ is a formula over $\mathcal{S}+$ with $\mathrm{FV}(\theta) = \{x\}$. Then for some $i \in \mathbb{N}$, $\phi_i = \neg(\forall x)\theta$. By the construction, ψ_i is a witness for θ. □

Proof of a special case of the Model Existence Theorem.

Proof of the Model Existence Theorem for $\mathcal{S}$ that has no function nor constant symbols. By Lemma 1.6, there is $\Gamma \subseteq \Gamma_1$ over $\mathcal{S}+$ that is consistent and contains witnesses. Let $\bar{\Gamma}$ be the set of all sentences that can be proven from Γ_1 (in particular, $\bar{\Gamma}$ contains all the axioms over $\mathcal{S}+$). Notice that $\bar{\Gamma}$ is still consistent (If $\bar{\Gamma} \vdash \bot$ then $\Gamma_1 \vdash \bot$ with the same proof.)

We construct a model $\mathcal{A}$ of $\bar{\Gamma}$. The domain A is the set of witnesses, i.e., the different constants $c_1, c_2, \ldots$ in $\mathcal{S}+$. (If $\bar{\Gamma} \vdash (c_i = c_j)$, we declare the constants c_i and c_j to be the same element of A.) We set ${c_i}^{\mathcal{A}} = c_i$.

$\bar{\Gamma}$ is consistent in the sense of propositional logic. Hence $\bar{\Gamma}$ admits a model V in the sense of propositional logic:

$$V\colon \{propositions\} \to \{0, 1\}$$

is an assignment such that for every proposition $\phi \in \bar{\Gamma}$, $V(\phi) = 1$.

In particular, if R^k_l is a relation symbol and $d_1, \ldots, d_k$ are elements of A then $R^k_l(d_1, \ldots, d_k)$ is a sentence (over $S+$) and V assigns it a truth value 0 or 1. We interpret

$$R_l^{k\mathcal{A}} = \{(d_1, \ldots, d_k) \in A^k \mid V(R^k_l(d_1, \ldots, d_k)) = 1\}.$$

1.8. **Claim.** *For every sentence ϕ,*

$$\mathcal{A} \models \phi \text{ iff } V(\phi) = 1.$$

Proof of the claim. The proof is by induction on ϕ.

(1) If ϕ is atomic then $\phi = R^k_l(c_1, \ldots, c_k)$. In this case

$$A \models \phi \Leftrightarrow (c_1, \ldots, c_k) \in R^k_l \Leftrightarrow V(R^k_l(c_1, \ldots, c_k)) = 1 \Leftrightarrow V(\phi) = 1.$$

(2) If $\phi = (\phi_1 \to \phi_2)$, then by definition of $\models$,

$$\mathcal{A} \models \phi \Leftrightarrow \text{ it is not true that } (\mathcal{A} \models \phi_1 \text{ and } \mathcal{A} \not\models \phi_2),$$

if and only if (by the induction assumption),

$$\text{it is not true that } V(\phi_1) = 1 \text{ and } V(\phi_2) = 0 \Leftrightarrow V(\phi) = 1.$$

(3) If $\phi = \bot$, then, by definition (of an assignment and of a model), $V(\phi) = 0$ and $\mathcal{A} \not\models \phi$.

(4) If $\phi = (\forall x)\psi$ with $\mathrm{FV}(\psi) = \{x\}$, we need to show
   (a) If $V(\phi) = 1$ then $\mathcal{A} \models \phi$.
   (b) If $\mathcal{A} \models \phi$ then $V(\phi) = 1$.

   To show part (a), we show that for every constant symbol c_n, $n \in \mathbb{N}$, we have $V(\psi[\frac{c_n}{x}]) = 1$. Indeed, we assume that $\bar{\Gamma}$ includes all the axioms, so $(\phi \to \psi[\frac{c_n}{x}]) \in \bar{\Gamma}$ hence $V(\phi \to \psi[\frac{c_n}{x}]) = 1$. Since $V(\phi) = 1$ and by the truth table of $\to$ we get $V(\psi[\frac{c_n}{x}]) = 1$. Hence, by the induction assumption,

$\mathcal{A} \models \psi[\frac{c_n}{x}]$. In other words, for every element $a \in A$, $\mathcal{A} \models \psi(\frac{a}{x})$. By definition of $\models$ (for a sentence that begins with $(\forall x)$), $\mathcal{A} \models (\forall x)\psi$, i.e., $\mathcal{A} \models \phi$.

To prove part (b), we will suppose that $V(\phi) = 0$ and show that $\mathcal{A} \not\models \phi$. Since $V((\forall x)\psi) = 0$, the value $V(\neg(\forall x)\psi) = 1$. Since $\bar{\Gamma}$ contains witnesses, there is a witness c_n such that $(\neg(\forall x)\psi \rightarrow \neg\psi[\frac{c_n}{x}]) \in \bar{\Gamma}$. Then $V(\neg\psi[\frac{c_n}{x}]) = 1$. By the induction assumption, this implies $\mathcal{A} \models \neg\psi[\frac{c_n}{x}]$ hence (by definition of $\models$) $\mathcal{A} \not\models (\forall x)\psi$.

$\square$

In particular, if $\phi \in \Gamma \subseteq \bar{\Gamma}$ then $V(\phi) = 1$ hence (by the claim) $A \models \phi$. Hence the restriction of $\mathcal{A}$ to the symbol set $\mathcal{S}$ of Γ is a model of Γ. $\square$