18.510: INTRODUCTION TO MATHEMATICAL LOGIC AND SET THEORY, FALL 08

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1. Predicate Calculus, First-order Logic

Recall the symbol set

$$\mathcal{S}_{\text{ar}} := \{+, \cdot, 0, 1\}$$

where $+$ and $\cdot$ are binary function symbols, and 0 and 1 are constants.

Also recall the $\mathcal{S}_{\text{ar}}$-structure

$$\mathcal{N} := (\mathbb{N}, +^{\mathcal{N}}, \cdot^{\mathcal{N}}, 0^{\mathcal{N}}, 1^{\mathcal{N}})$$

where $+^{\mathcal{N}}$ and $\cdot^{\mathcal{N}}$ are the usual addition and multiplication on $\mathbb{N}$ and $0^{\mathcal{N}}$ and $1^{\mathcal{N}}$ are the numbers zero and one, respectively.

In this lecture, we consider $\mathcal{N}$ as a structure of the symbol set $(0, +, \cdot, s)$, where we interpret s as the successor function $s(n) = n + 1$ for $n \in \mathbb{N}$.

1.1. *Example.* The structure $\mathcal{N}$ satisfies the so-called *Peano axiom system* PA:

(1) $(\forall x_1)(\forall x_2)((s(x_1) = s(x_2)) \to (x_1 = x_2))$ (i.e., s is one-to-one).
(2)
- $(\forall x)(\neg(x = 0) \to (\exists y)(s(y) = x))$ (i.e., every element in $\mathbb{N}$ that is not 0 is a successor of another element).
- $\neg(\exists y)(s(y) = 0)$ (i.e., there is no element of $\mathbb{N}$ such that 0 is its successor).

(3)
- $(\forall y)(0 + y = y)$
- $(\forall x)(\forall y)(s(x) + y = s(x + y)$

(4)
- $(\forall y)(0 \cdot y = 0)$
- $(\forall x)(\forall y)(s(x) \cdot y = x \cdot y + y)$

(5) Induction principle: for all $x, y_1, \ldots, y_n$ and ϕ with $\mathrm{FV}(\phi) = \{x, y_1, \ldots, y_n\}$,

$$(\forall \bar{y})([\phi(0, \bar{y}) \wedge (\forall z)(\phi(z, \bar{y}) \to \phi(s(z), \bar{y}))] \to (\forall x)\phi(x, \bar{y})),$$

where $\bar{y} = \{y_1, \ldots, y_n\}$ and $(\forall \bar{y}) = (\forall y_1) \ldots (\forall y_n)$.

The theory of $\mathcal{N}$ and non-standard models. For an S-structure $\mathcal{A}$, we denote by $\mathrm{Th}(\mathcal{A})$ the set of the sentences ψ in the first-order language $L(S)$ such that $\mathcal{A} \models \psi$. We call $\mathrm{Th}(\mathcal{A})$ the (first-order) *theory* of $\mathcal{A}$.

1.2. **Theorem.** *There is a model of* $\mathrm{Th}(\mathcal{N})$ *that is not isomorphic to* $\mathcal{N}$.

Let $\mathcal{A}$ be a model of $\mathrm{Th}(\mathcal{N})$ over $\mathcal{S} = \{0, +, \cdot, s\}$. Define $s^1(x) = s(x)$, $s^2(x) = s(s(x))$, ..., in general, $s^{n+1}(x) = s(s^n(x))$. The domain A contains the elements $0^{\mathcal{A}}$, $s^{\mathcal{A}}(0^{\mathcal{A}})$, $s^{2\mathcal{A}}(0^{\mathcal{A}})$, ... Does A contain other elements? such elements are called *non standard* elements. If A contains non-standard elements, then $\mathcal{A}$ is called a *non standard* model of $\mathrm{Th}(\mathcal{N})$. In the proof of Theorem 1.2, we find a non standard model of $\mathrm{Th}(\mathcal{N})$.

Proof. We add a constant c to $\mathcal{S}$ to get the symbol set $\{c, 0, +, \cdot, s\}$. We set

$$\Gamma^* = \{c \neq 0,\, c \neq s(0),\, c \neq s^2(0),\, \ldots\}$$

and

$$\Gamma = \mathrm{Th}(\mathcal{N}) \cup \Gamma^*.$$

1.3. **Claim.** Γ *admits a model.*

Proof. By the compactness theorem, it is enough to show that every finite subset of Γ admits a model. Indeed, a finite subset $\Gamma_0 \subset \Gamma$ is contained in $\mathrm{Th}(\mathcal{N}) \cup \{c \neq 0, \ldots, c \neq s^n(0)\}$; Define a structure $\mathcal{A} = \mathcal{A}_{n+1}$ as follows:

- $A = \mathbb{N}$,
- $+^{\mathcal{A}} = +^{\mathcal{N}}$,
- $\cdot^{\mathcal{A}} = \cdot^{\mathcal{N}}$,
- $s^{\mathcal{A}} = s^{\mathcal{N}}$,
- $0^{\mathcal{A}} = 0^{\mathcal{N}}$,
- $c^{\mathcal{A}} = n + 1$.

Every sentence in $\mathrm{Th}(\mathcal{N})$ is true in $\mathcal{N}$, and does not use c, hence it is true in $\mathcal{A}$. By the choice of $c^{\mathcal{A}}$, the $n+1$ first sentences of Γ^* are also true in $\mathcal{A}$. $\square$

Thus, there is a model $\mathcal{B}'$ of Γ. Let $\mathcal{B}$ be the structure with the same domain and interpretation for $(+, \cdot, s, 0)$ as $\mathcal{B}'$ but without the interpretation for c, Then $\mathcal{B}$ is a model of $\mathrm{Th}(\mathcal{N})$ whose domain contains the non-standard element $c^{\mathcal{B}'}$. $\square$

Notice that a model of $\mathrm{Th}(\mathcal{N})$ is in particular a model of PA $\subseteq \mathrm{Th}(\mathcal{N})$. Hence Theorem 1.2 implies that there is a model of PA that is not isomorphic to $\mathcal{N}$.

Theorem 1.2 gives an example of two structures (of the same symbol set) that satisfy the same sentences but are not isomorphic, (as promised when we talked about isomorphism of structures).

Godel's enumeration.

1.4. **Lemma.** *Assume that the symbol set $\mathcal{S}$ is countable. Then the set of all formulas is countable, i.e., can be listed as $\phi_1, \phi_2, \ldots$ such that every formula appears as ϕ_i for some $i \in \mathbb{N}$.*

Proof. To each formula ϕ in $\mathcal{L} = \mathcal{L}(S)$ we assign a natural number $\sharp\phi$, the *Godel number* of ϕ, such that $\phi \mapsto \sharp\phi$ is one-to-one: if $\phi \neq \phi'$ then $\sharp\phi \neq \sharp\phi'$. Given such an enumeration we define ϕ_m to be the formula with Godel number m, if exists, and $\perp$ if not. This gives a well defined onto map $\mathbb{N} \to \mathcal{L}$, i.e., the set of formulas is countable.

We assign Godel numbers by induction on ϕ. First we assign Godel numbers to the alphabet:

(1) to the logic symbols

(	)	$\to$	$\perp$	$\forall$	$=$
$2^1 \cdot 3$	$2^1 \cdot 3^2$	$2^1 \cdot 3^3$	$2^1 \cdot 3^4$	$2^1 \cdot 3^5$	$2^1 \cdot 3^6$

(2) to each variable symbol v_k: $\sharp v_k = 2^2 \cdot 3^k$.
(3) to each relation symbol R_i^k: $\sharp R_i^k = 2^3 \cdot 3^{i+1} \cdot 5^{k+1}$.
(4) to each function symbol F_j^m: $\sharp F_j^m = 2^4 \cdot 3^{j+1} \cdot 5^{m+1}$.
(5) to each constant symbol c_j: $\sharp c_j = 2^5 \cdot 3^{j+1}$.

Then we assign Godel numbers to the terms by induction on the construction of the term. If $t = F_j^m(t_1, \ldots, t_m)$ then

$$\sharp t = 2^6 \cdot 3^{\sharp F_j^m} \cdot 5^{\sharp t_1} \cdot 7^{\sharp t_2} \cdot \ldots \cdot p_{m+2}{}^{\sharp t_m},$$

where p_i is the i-th prime number.

We assign Godel numbers to the formulas by induction on the construction of the formula:

(1) $\sharp(t_1 = t_2) = 2^7 \cdot 3^{\sharp t_1} \cdot 5^{\sharp t^2}$;
(2) $\sharp(R_i^k(t_1, \ldots, t_k)) = 2^8 \cdot 3^{\sharp R_i^k} \cdot 5^{\sharp t_1} \cdot \ldots \cdot p_{k+2}^{\sharp t_k}$;
(3) $\sharp(\psi_1 \to \psi_2) = 2^9 \cdot 3^{\sharp\psi_1} \cdot 5^{\sharp\psi_2}$;
(4) $\sharp(\forall x_n)\psi = 2^{10} \cdot 3^{\sharp x_n} \cdot 5^{\sharp\psi}$.

The fact that the assignment is one-to-one follows from the unique decomposition of natural numbers to prime factors.

$\square$

1.5. *Remark.* We can continue the proof of Lemma 1.4 and assign unique Godel numbers to proofs: $\sharp(\alpha_1, \dots, \alpha_k) = 2^{11} \cdot 3^{\sharp\alpha_1} \cdot 5^{\sharp\alpha_2} \cdot \dots \cdot p_{k+1}^{\sharp\alpha_k}$. Thus the set of all proofs over $\mathcal{S}$ is countable.

Godel's Theorem and Tarski's Theorem.

1.6. **Theorem** (Godel's Theorem)**.** *There are formulas ϕ_i, $i = 1, 2, 3, 4$ with $\mathrm{FV}(\phi_i) = \{x\}$ for $i = 1, 2, 4$ and $\mathrm{FV}(\phi_3) = \{x, y\}$, such that:*

(1) $\mathcal{N} \models \phi_1(n) \Leftrightarrow n$ *is the Godel number of a sentence*;
(2) $\mathcal{N} \models \phi_2(n) \Leftrightarrow n$ *is the Godel number of a proof from* PA;
(3) $\mathcal{N} \models \phi_3(n, m) \Leftrightarrow n$ *is the Godel number of a proof of ϕ with $\sharp\phi = m$ from* PA;
(4) $\phi_4 = (\exists x)\phi_3(x, y)$, $\mathcal{N} \models \phi_4(m) \Leftrightarrow \exists n \in \mathbb{N}$ *such that n is the Godel number of a proof of ϕ with $\sharp\phi = m$ from* PA .

1.7. **Theorem** (Tarski's Theorem)**.** *There is no formula ψ (with* $\mathrm{FV}(\psi) = \{x\}$*) such that $\mathcal{N} \models \psi(n) \Leftrightarrow n$ is the Godel number of a sentence that is true in $\mathcal{N}$.*

1.8. **Corollary.** *It is not true that $\mathcal{N} \models \phi$ iff* $\mathrm{PA} \vdash \phi$.

Hence, using the completeness theorem, there is a sentence ϕ such that $\mathcal{N} \models \phi$ but $\mathrm{PA} \nvDash \phi$.

Proof of Tarski's Theorem. A subset $A \subseteq \mathbb{N}$ is called *definable* if there is a formula ϕ with $\mathrm{FV}(\phi) = \{x\}$ such that $\phi^{\mathcal{N}} = A$, i.e.,

$$n \in A \Leftrightarrow \mathcal{N} \models \phi[\frac{s^n(0)}{x}].$$

For example, the set A of even numbers is definable by the formula $\phi(x) = (\exists y)(y + y = x)$.

Similarly, a subset $A \subseteq \mathbb{N}^k$ is *definable* if there is a formula ϕ with $\mathrm{FV}(\phi) = \{x_1, \dots, x_k\}$ such that $\phi^{\mathcal{N}} = A$.

To construct an example of a non-definable subset of $\mathbb{N}$, list all the formulas ϕ with $\mathrm{FV}(\phi) = \{x\}$ using Godel's enumeration, (see Lemma 1.4), as $\phi_1, \phi_2, \dots$ We will build the example according to the "opposite" of the diagonal in the following table.

	0	1	2	…
ϕ_0	0	1	0	…
ϕ_1	1	1	0	…
ϕ_2	1	0	0	…
⋮	⋮	⋮	⋮	

(We write $\phi_i(j) = 1$ iff $\mathcal{N} \models \phi_i(j)$, where j stands for $s^j(0)$.)

Set

$$\Delta = \{n \in \mathbb{N} \,|\, \mathcal{N} \models \neg\phi_n(n)\}.$$

1.9. **Claim.** *The set Δ is not definable.*

Proof. Suppose that Δ is definable, i.e., there is a formula ϕ with $\mathrm{FV}(\phi) = \{x\}$ such that $\Delta = \phi^{\mathcal{N}}$. The formula ϕ is one of the formulas in the list, $\phi = \phi_k$ for some $k \in \mathbb{N}$. However, for this number k,

$$k \in \Delta \Leftrightarrow \mathcal{N} \models \neg\phi_k(k) \Leftrightarrow k \text{ is not in } {\phi_k}^{\mathcal{N}} = \phi^{\mathcal{N}},$$

hence $\Delta \neq \phi^{\mathcal{N}}$ in contradiction with our assumption. □

1.10. **Corollary.** *The set $\{n \in \mathbb{N} \,|\, \mathcal{N} \models \psi_n\}$, where $\psi_n = \neg\phi_n(n) = \neg\phi_n[\frac{s^n(0)}{x}]$, is not definable.*

1.11. **Lemma.** *The function*

$$h \colon \mathbb{N} \to \mathbb{N}$$

mapping $n \mapsto \sharp\psi_n$ is definable, i.e., the set $(n, \sharp\psi_n)$ is a definable subset of $\mathbb{N}^2$.

1.12. **Corollary.** *The set*

$$T = \{n \in \mathbb{N} \,|\, \mathcal{N} \models \text{ the sentence with Godel number } n\}$$

is not definable.

Proof. Suppose that T is definable by a formula α, i.e.,

$$n \in T \Leftrightarrow \mathcal{N} \models \alpha(s^n(0)).$$

By Lemma 1.11, h is definable by a formula β, i.e.,

$$h(n) = m \Leftrightarrow \mathcal{N} \models \beta(s^n(0), s^m(0)).$$

Then

$$n \in \Delta \Leftrightarrow \mathcal{N} \models \neg\phi_n(s^n(0)) \Leftrightarrow \mathcal{N} \models \psi_n \Leftrightarrow h(n) \in T \Leftrightarrow \mathcal{N} \models \alpha(h(n))$$
$$\Leftrightarrow \mathcal{N} \models (\exists y)(\beta(s^n(0), y) \wedge \alpha(y)).$$

Hence, Δ is also definable, in contradiction with Claim 1.9. □

This proves Tarski's Theorem. In other words, there is no inner truth definition in $\mathcal{N}$.