Problem 1. Consider the function \( f \), with domain \((-\infty, \infty)\), given by
\[
 f(x) = (x^4 + 1)^{1/2}.
\]

(1.a) Why does there not exist an inverse of \( f \)?

In order for a function to have an inverse, it must be one-to-one on its domain.
\( f \) is not one-to-one. For example, \( f(1) = f(-1) = 2 \)

(1.b) Give an example of an interval \( D = [t_1, t_2] \) such that the restriction of \( f \) to the domain \( D \) admits an inverse.

\([0, 20]\). In fact any closed interval that is a subset of \((0, \infty)\) would work.

(1.c) Find an expression for \( f^{-1} \) when the domain of \( f \) is restricted to the interval you gave in part (1.b).

Say \( y = (x^4 + 1)^{1/2} \).
\[
x = (y^2 - 1)^{1/4}
\]
\[
y^2 = x^4 + 1
\]
\[
x = (y^2 - 1)^{1/4}
\]

Problem 2. For each of the following, give the value of the indicated limit if that limit exists, or state that the limit does not exist.

(2.1) the limit as \( x \) goes to 0 of
\[
 x \cdot |x| \quad \lim_{x \to 0} x \cdot |x| = 0
\]

(2.2) the limit as \( x \) goes to 0 of
\[
 \frac{|x|}{x} \quad \text{does not exist (right and left hand limits are different)}
\]

(2.3) the limit as \( x \) goes to 0 of
\[
 \frac{\cos \left( \frac{x^2 + x}{3} \right) + \sin(4x)}{x} \quad \lim_{x \to 0} \frac{\cos \left( \frac{x^2 + x}{3} \right) + \sin(4x)}{x} = \cos(0) + \sin(0) = 1 + 0 = 1
\]

(2.4) the limit as \( h \) goes to 0 of
\[
 \frac{(x + h)^2 - x^2}{h} \quad \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} 2x + h = 2x
\]
Quiz 3

Problem 1. Consider the function

\[ f(x) = \begin{cases} 
0 & \text{if } x = k\pi \text{ for some integer } k \\
\frac{\sin(x)}{|\sin(x)|} & \text{otherwise} 
\end{cases} \]

(1 pt)(1.a) What is the domain of \( f \)?
\((-\infty, \infty)\)

(1 pt)(1.b) At which points does \( f \) have left-hand limits?
Every point in \((-\infty, \infty)\)

(1 pt)(1.c) At which points does \( f \) have right-hand limits?
Every point in \((-\infty, \infty)\)

(3 pts)(1.d) At which points does \( f \) have a limit?
Every point except those of the form \( x = k\pi \) for integers \( k \).

Problem 2. True or False (circle the correct option) (1 pt each).

(2.1) If \( f \) is undefined on the entire interval \([a, b]\), it can still be the case that
\[ \lim_{x \to b^-} f(x) \]
exists. \( \bigcirc \)
\( T \) \( F \)

(2.2) If \( \lim_{x \to x_0^+} f(x) = L \) and \( \lim_{x \to x_0^-} f(x) = M \), then for \( g(x) \) defined by the following,
\[ g(x) = \begin{cases} 
 f(x) - M & \text{for } x < x_0 \\
 f(x) - L & \text{otherwise} 
\end{cases} \]
we have that the limit \( \lim_{x \to x_0} f(x) \) exists, and is 0.
\( \bigcirc \)
\( T \ \text{(always)} \) \( F \ \text{(cannot be determined)} \)

(2.3) If the left hand limit of \( f \) as \( x \) approaches \( x_0 \) is negative and the right hand limit of \( f \) as \( x \) approaches \( x_0 \) is positive then the limit of \( f \) cannot exist as \( x \) approaches \( x_0 \).
\( \bigcirc \)
\( T \) \( F \)

(2.4) If \( f \) does not have a left hand limit as \( x \) approaches \( x_0 \), and the same is true for
\( g \), it can still happen that \( f + g \) has a left hand limit as \( x \) approaches \( x_0 \).
\( \bigcirc \)
\( T \) \( F \) (For example, let \( g = -f \). Then \( (f + g)(x) = 0 \) for all \( x \).)
Problem 1.

(4 pts) For what values of $a$ and $b$ is the function

$$f(x) = \begin{cases} 
-2, & x \leq -1 \\
ax - b, & -1 < x < 1 \\
3, & x \geq 1 
\end{cases}$$

continuous?

If $f$ is continuous, we need the line $ax + b$ to pass through $(-1, -2)$ and $(1, 3)$.

The slope of such a line is

$$\frac{3 - (-2)}{1 - (-1)} = \frac{5}{2}.$$

If $y = mx + B$, and $m = \frac{5}{2}$,

Then $3 = \frac{5}{2} \cdot 1 + B$, so $B = \frac{1}{2}$. So $y = \frac{5}{2}x + \frac{1}{2}$.

\[
a = \frac{5}{2}, \quad b = -\frac{1}{2}
\]

Problem 2.

(2 pts) (a) Graph

\[
f(x) = \begin{cases} 
1 - x^2, & x \neq 1 \\
2, & x = 1 
\end{cases}
\]

\[
\text{(in some white space to the right)}
\]

(2 pts) (b) Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$

\[
\lim_{x \to 1^+} f(x) = 0, \quad \lim_{x \to 1^-} f(x) = 0
\]

(2 pts) (c) Does $\lim_{x \to 1} f(x)$ exist? If so what is it? If not, why not?

Yes, $\lim_{x \to 1^+} f(x) = 0$. 

Quiz 5

NAME: 

INSTRUCTOR:

Problem 1. Write the letter of the graph (X, Y, Z, or W) which best represents the graph of the derivative of the function \( f \).

1.a) (1 point) A stopped car backs up and then goes forward. \( f \) is the position of the car relative to its starting location. ("forward" is the positive direction.) \( \text{Z} \)

1.b) (1 point) I throw a ball up in the air. \( f \) is the height of the ball. \( \text{X} \)

1.c) (1 point) A colony of bacteria is growing. \( f \) is the number of bacteria in the colony. \( \text{W} \)

1.d) (1 point) The value of a stock steadily increases and then remains constant. \( f \) is the value of the stock. \( \text{Y} \)

Problem 2. (2 points) Let \( f(x) = 4x^2 + x \). Use the definition of the derivative of \( f \) at \( x_0 \),

\[
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},
\]

to find an expression for \( f'(x) \).

\[
f'(x) = \lim_{h \to 0} \frac{4(x_0 + h)^2 + (x_0 + h) - 4x_0^2 - x_0}{h} = \lim_{h \to 0} \frac{4x_0^2 + 8x_0h + 4h^2 - 4x_0^2 - x_0 + x_0 + h}{h} = \lim_{h \to 0} 8x_0 + 4h + 1 = 8x_0 + 1
\]

\[
f'(x) = 8x + 1
\]

Problem 3. (2 points) Let \( g(x) = 2x^{500} + e^x + \pi \). Find an equation for the line tangent to \( g \) at the point \( (0, g(0)) \).

\[
g'(x) = 1000 \cdot x^{499} + e^x
\]

\[
g'(0) = 1, \\
\]

\[
g(0) = e^0 + \pi = 1 + \pi.
\]

\[
y = x + 1 + \pi
\]

is the line tangent to \( g \) at \( (0, g(0)) \).