A Game of Life on Penrose Tilings

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Outline

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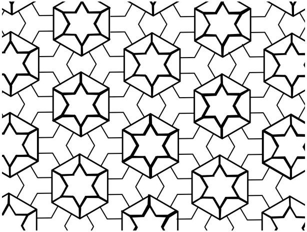
Is there a way to encode/process information in a nonperiodic environment?

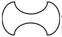
A **tiling** of the plane is a partition of the plane into sets (called **tiles**) that are topological disks.

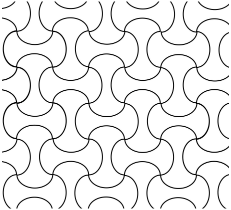
An *n*-**hedral** tiling – there are *n* different types of tiles, i.e. every tile is congruent to one of *n* fixed subsets of the plane, called **prototiles**.

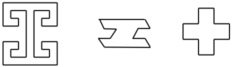


ex. a 4-hedral tiling

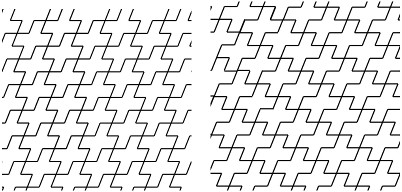
How many tilings can you make?

How many distinct tilings does a given set of prototiles admit?
A prototile is called **n-morphic** if it admits precisely *n* distinct monohedral tilings of the plane.
For example, this tile  is monomorphic; it is clear that there is only one way to tile the plane with it.



More examples of monomorphic prototiles: 

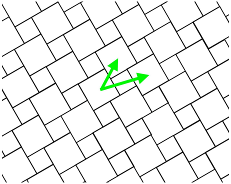
Ex. two tilings admitted by a dimorphic prototile:



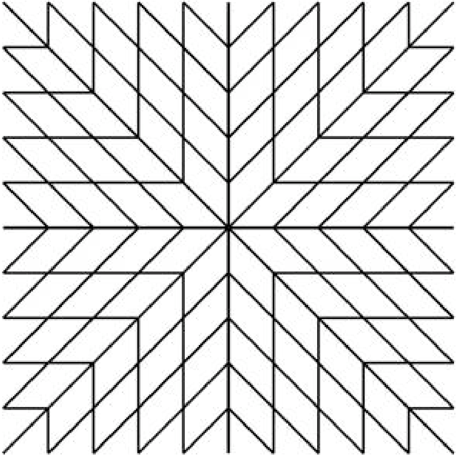
**Open question: for every $n \in \mathbb{N}$, does there exist a *n*-morphic prototile?

**Open question: does there exist a prototile (or set of prototiles) that admits countably but not uncountably many distinct tilings?

A tiling is **periodic** if its symmetry group includes translations in two nonparallel directions. Otherwise the tiling is **nonperiodic**.



nonperiodic tiling example



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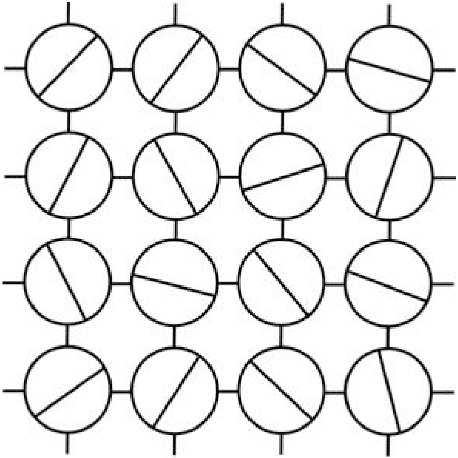

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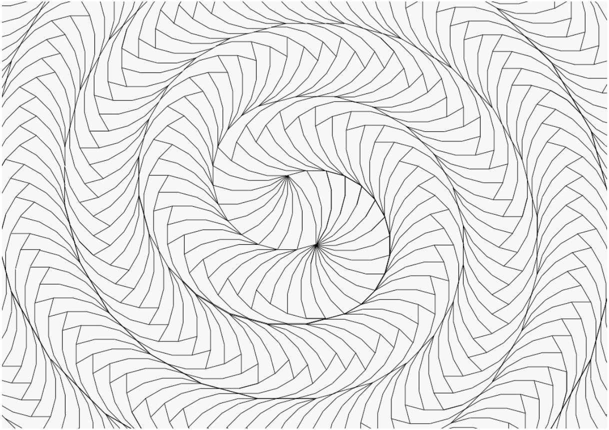

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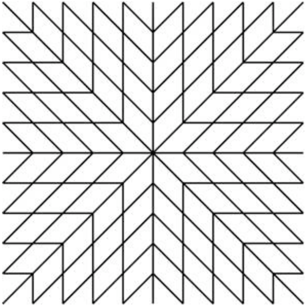

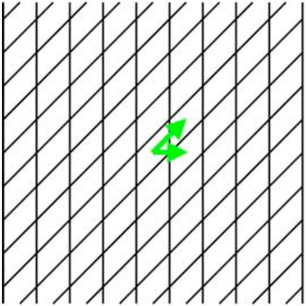

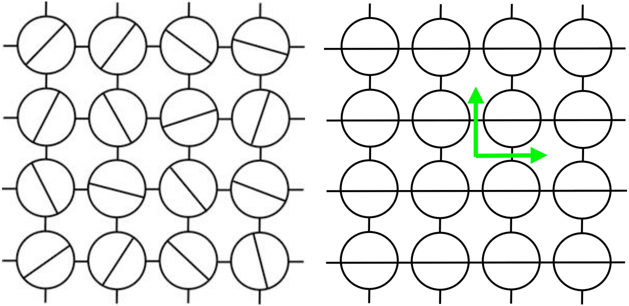

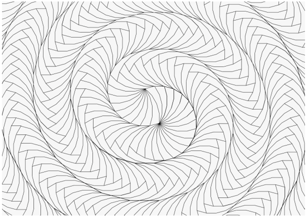
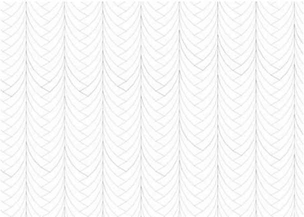

aperiodic tiling

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A set of prototiles with this property (they tile the plane but never periodically) is said to be **aperiodic** and a tiling admitted by an aperiodic set of prototiles is called an **aperiodic tiling**.

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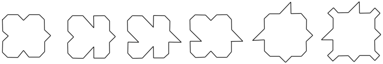
The (somewhat surprising) answer is yes! The first aperiodic set was constructed by Robert Berger in 1966 and used 20,426 prototiles!

The next obvious question is "Can we find a smaller set of aperiodic prototiles?" and, in particular, "What is the smallest number of prototiles necessary to tile the plane aperiodically?"

After Berger's discovery, various mathematicians considered this question and discovered sets of aperiodic prototiles with fewer and fewer prototiles.

One well-known set of six aperiodic prototiles was discovered by Robinson in 1971.

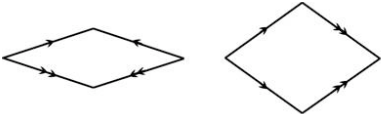


Penrose Tilings

The most famous example of aperiodic tilings, known as **Penrose tilings**, were discovered by Roger Penrose in the 1970s, and have only two prototiles.

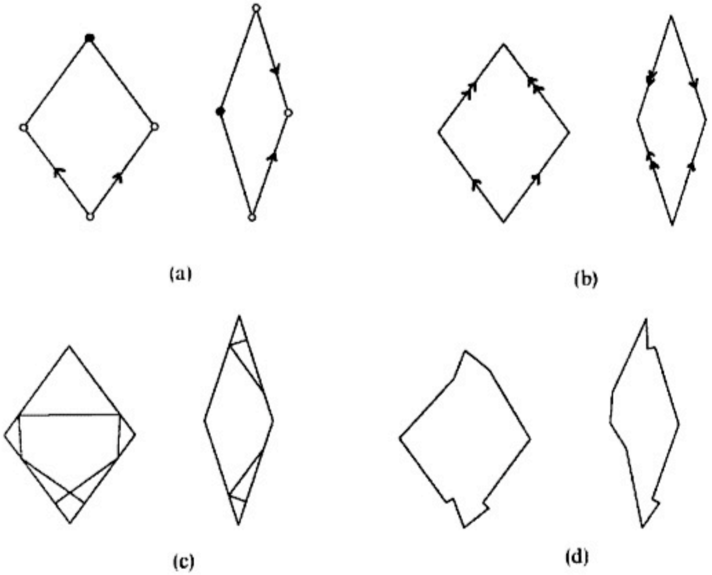
**Open Question: Does there exist a single prototile that tiles the plane aperiodically?

Here are the two prototiles in the aperiodic set discovered by Penrose:

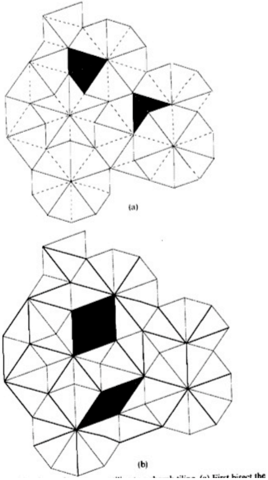


thin: angles $\pi/5$ and $4\pi/5$; thick: angles $2\pi/5$ and $3\pi/5$.

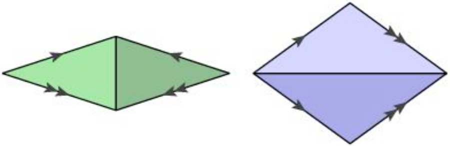
Equivalent representations of Penrose tilings



Equivalent representations - kites and darts

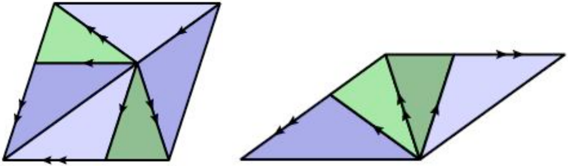


Inflation/Deflation



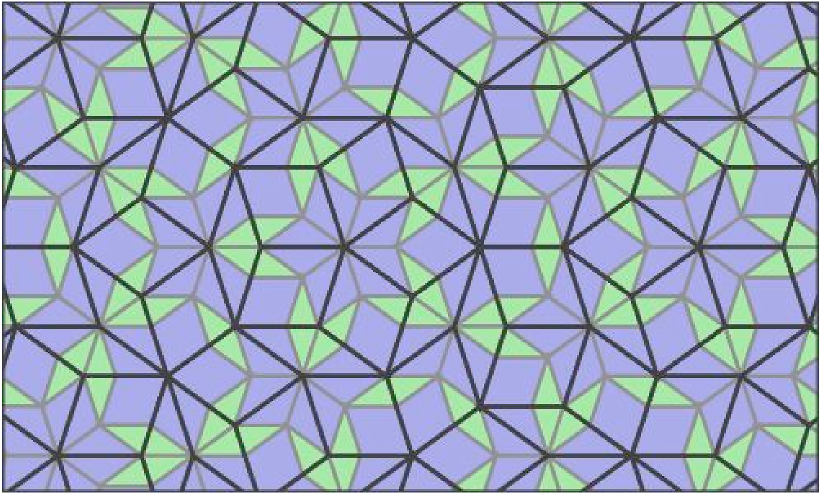
Prototiles look like

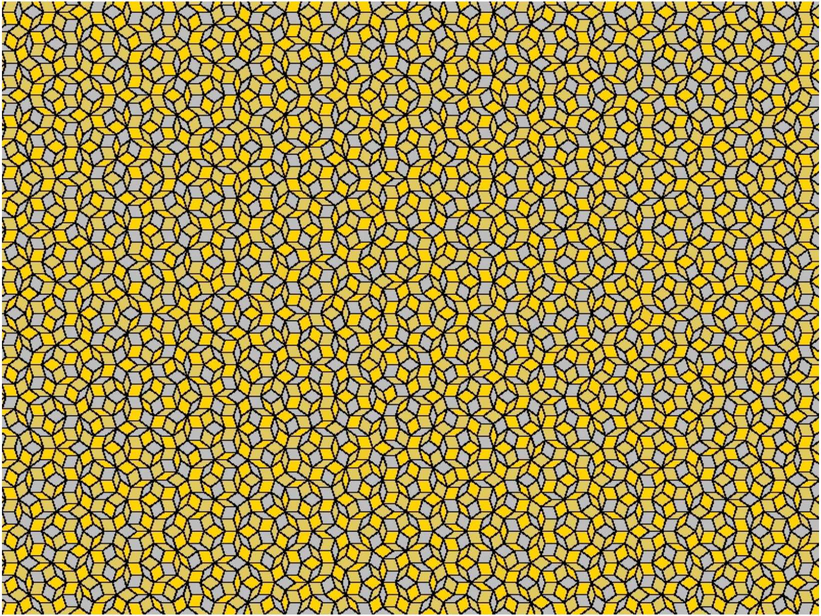
Inflation/Deflation Rules



1 thick $\rightarrow$ 2 thick and 1 thin; 1 thin $\rightarrow$ 1 thick and 1 thin.
(thin,thick): (0,1),(1,2),(3,5),(8, 13),(21, 34),(55,89),(144, 233)

Inflation/Deflation

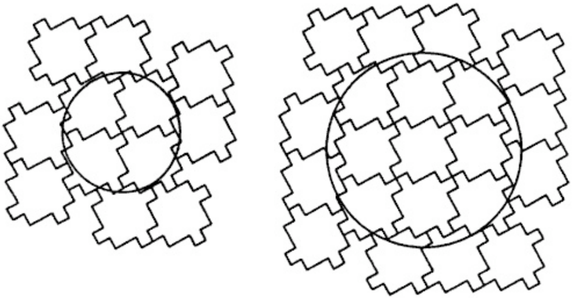




Extension Theorem

Theorem

Let $\mathcal{T}$ be any finite set of prototiles, each of which is a closed topological disk. If $\mathcal{T}$ tiles over arbitrarily large circular disks, then $\mathcal{T}$ admits a tiling of the plane.



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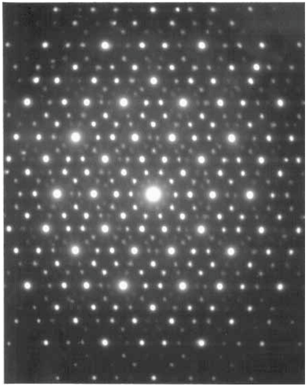
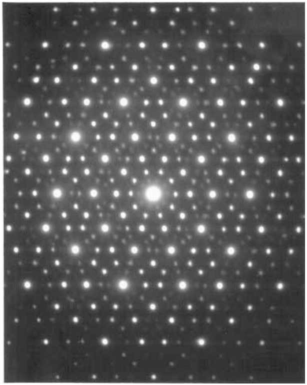
Suppose it is periodic. Then the ratio for the whole tiling equals the ratio in the fundamental unit, i.e. is rational. Therefore the Penrose tiling is nonperiodic.

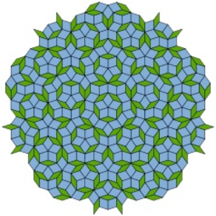
Penrose tilings:
-have no translational symmetries
-can have 5-fold rotations and/or reflections as symmetries.
-patch in a tiling is repeated infinitely many times in that tiling
-ratio of thick rhombs to thin rhombs = the golden ratio = 1.618...

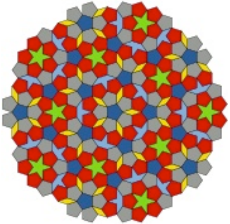
Quasicrystals

Before 1982, it was assumed that all solids that are "ordered" at the microscopic level were made from the periodic repetition of a fundamental unit. But in 1982 Schechtman cooled an aluminum-manganese sample and found that its diffraction patterns had 5-fold rotational symmetry.









The Crystallographic Restriction Theorem confines the rotational symmetries of translation lattices in two- and three-dimensional Euclidean space to orders 2, 3, 4, and 6.

So this “quasicrystal” is not periodic. But it is clearly “ordered.” Like Penrose tilings, can inflate/deflate.

Since then, many more examples of “quasicrystals” have been discovered.

Conway's Game of Life

Conway's Game of Life is a cellular automaton played on a regular grid of square "cells;" each cell of the automaton takes either state 0 or 1 ("living" or "dead") and updates its state in discrete time based on the states of its 8 closest neighbors, as follows:

-A dead cell with exactly three living neighbors becomes alive.
-A living cell with 2 or 3 living neighbors remains alive.
-In any other case the cell dies/remains dead.

Link to bitstorm

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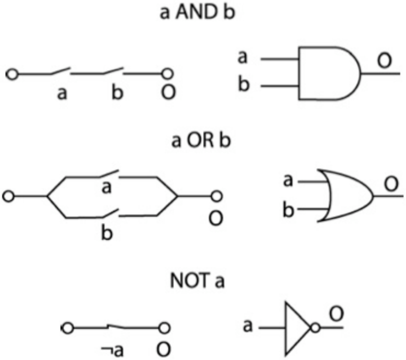
-self-replicating machines (i.e. patterns) exist. (DNA?)

Logic Gates

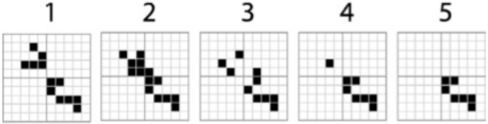
Every logical function, i.e. every possible result set of the combination of two Boolean variables, can be constructed using these three fundamental operators.



We then only have to implement AND, OR, NOT-gates to be able to manage any Boolean function. To implement a logical gate we therefore need:

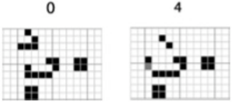
- Some kind of electrical pulses to represent inputs.
- Wires to transmit the electrical pulses.
- Processing devices which associate inputs and compute the Boolean result.
- A device placed after the processing device, able to check the output electrical pulses. This will represent the output.

An eater:



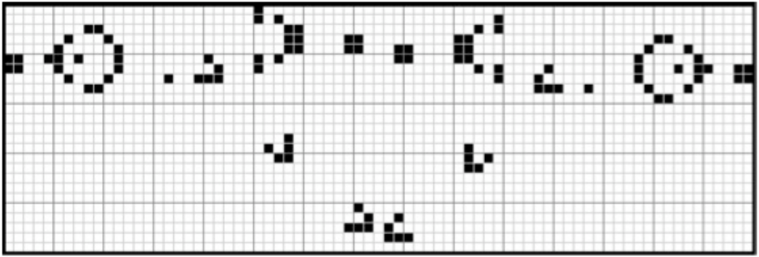
This consumes a glider and recovers to its original form.

An eater that detects gliders:

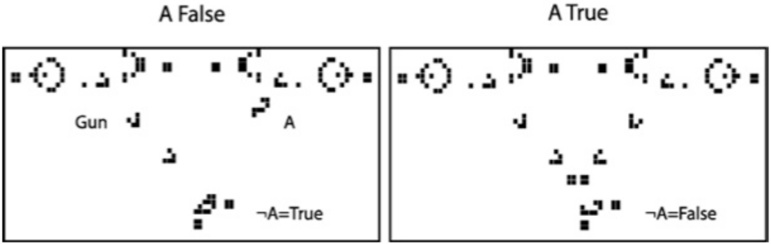


Output activation. The light colored cell is activated.

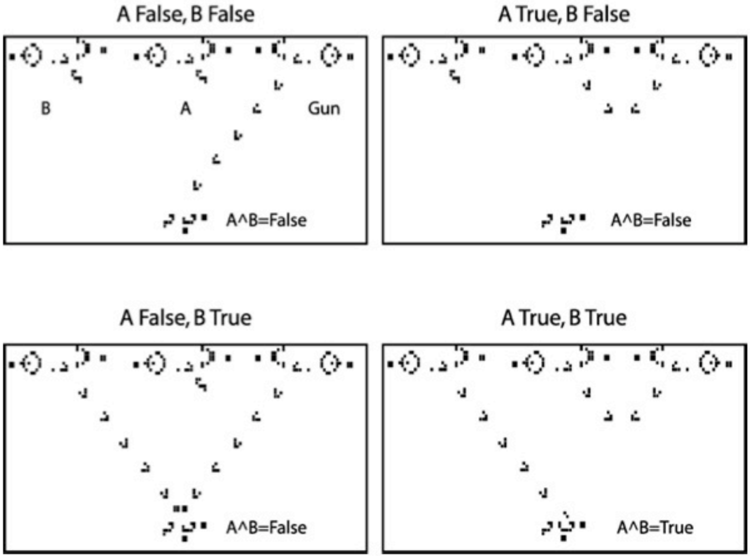
Glider gun streams annihilate each other if the distance between nascent gliders is even and there is a one-cell vertical offset between the glider streams.



NOT gate

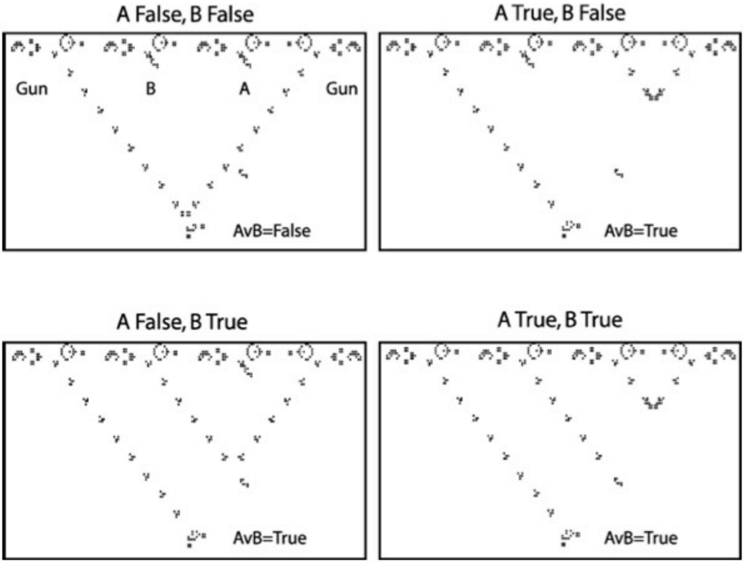


AND gate



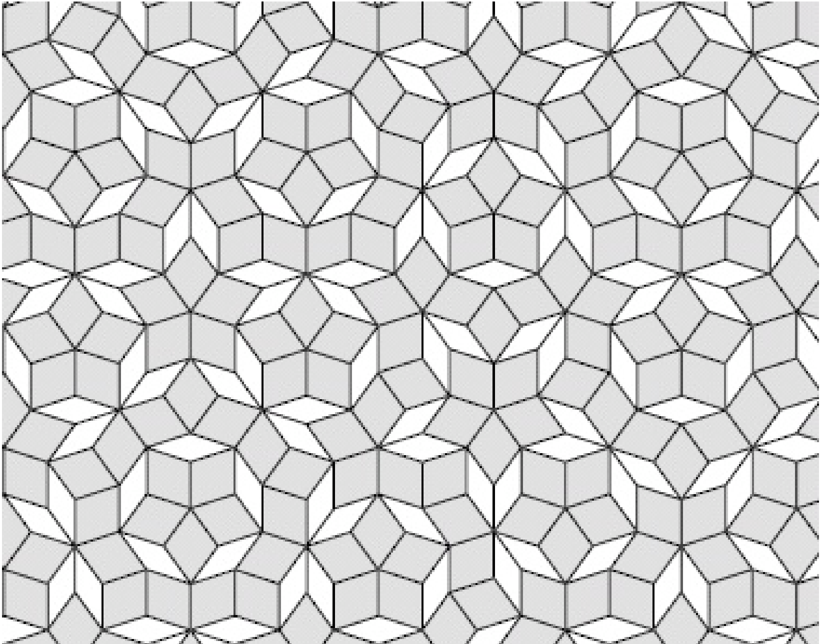
A False, B False
B
A
Gun
A^B=False
A True, B False
A^B=False
A False, B True
A^B=False
A True, B True
A^B=True

OR gate


A False, B False
Gun
B
A
Gun
AvB=False
A True, B False
AvB=True
A False, B True
AvB=True
A True, B True
AvB=True

Rennard's applet

Life...on a Penrose Tiling?

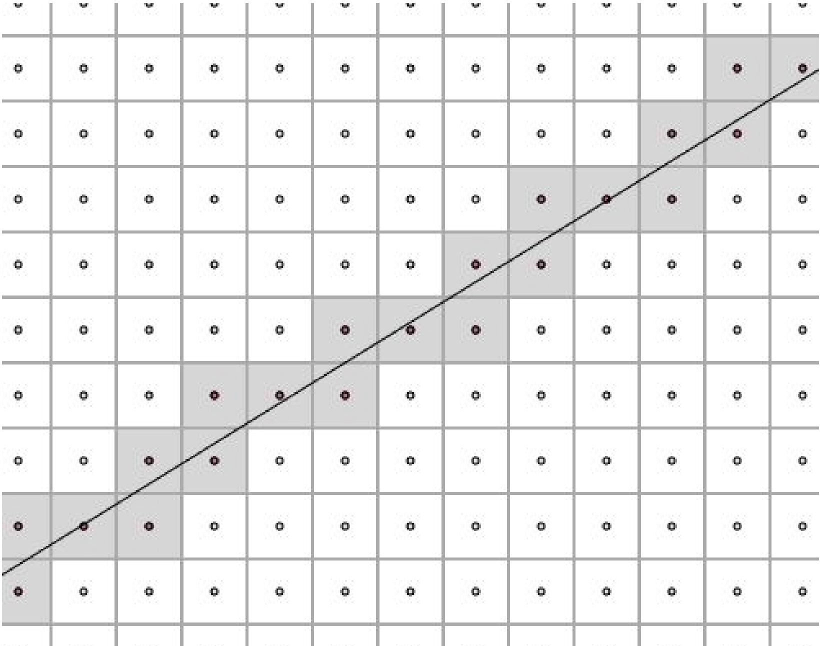


Previous research....

Various people tried lots of different sets of nearest-neighbors rules...but no gliders were found.

Ash and oscillators....

Projection method: 2 dimensions to 1



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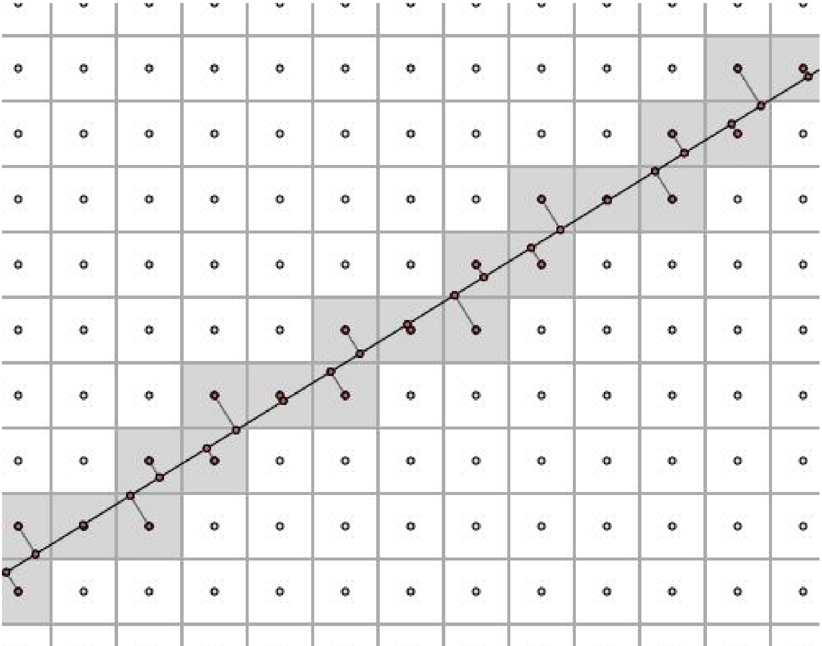

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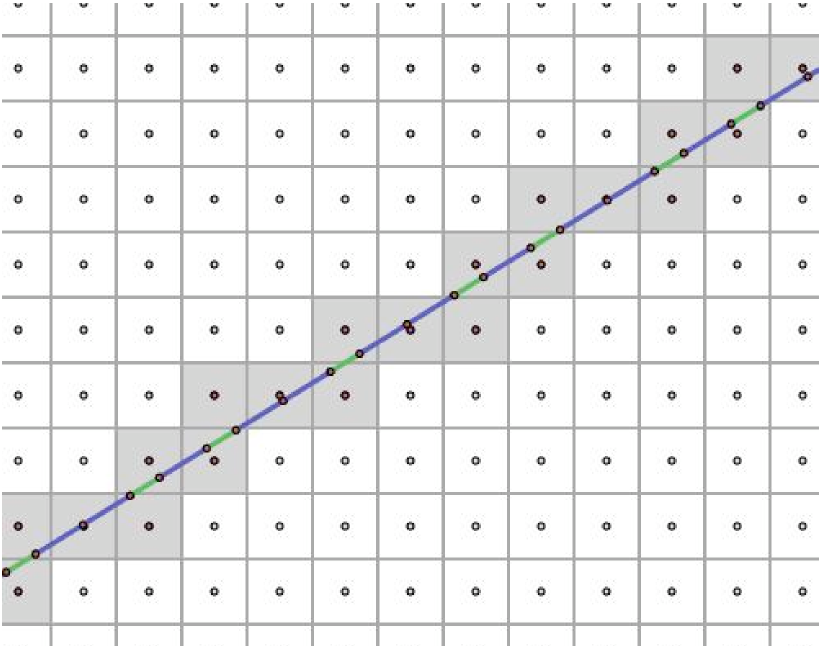


Penrose tilings: projection method

N. G. de Bruijn (1981). "Algebraic theory of Penrose's nonperiodic tilings of the plane, I, II" (PDF). Indagationes mathematicae 43 (No. 1): 3966.

de Bruijn's big result: Penrose tilings are obtained in an analogous way using a 2-dimensional plane cutting through 5-dimensional space at a (fixed) irrational “angle”.

The “angle” of the plane is fixed; you can pick the “offset” vector. de Bruijn: any offset vector so that the plane does not go through a “corner” gives rise to a Penrose tiling. Conversely, every Penrose tiling arises in this way.

The moral of the story: Penrose tilings are the "shadows" of the edges of 5-d cubes.
Faces of 5-d cubes $\rightarrow$ Penrose tiles (angle of face in relation to cutting plane determines whether a face becomes a thick rhomb or a thin rhomb).
Vertices $\rightarrow$ vertices

Generalize it: you can get aperiodic structures in 3-space as the projection of a regular/periodic structure in a higher dimension projected down to 3-space.

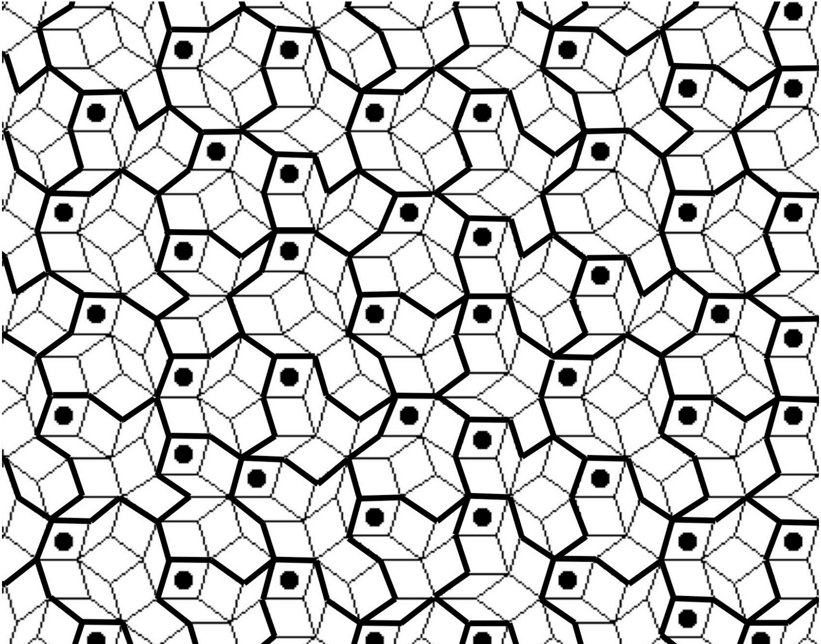
The Solution

The rough idea of the solution: play a "Game of Life" on $\mathbb{Z}^5$ (with appropriate rules) and project it down onto the plane, i.e. the Penrose tiling.

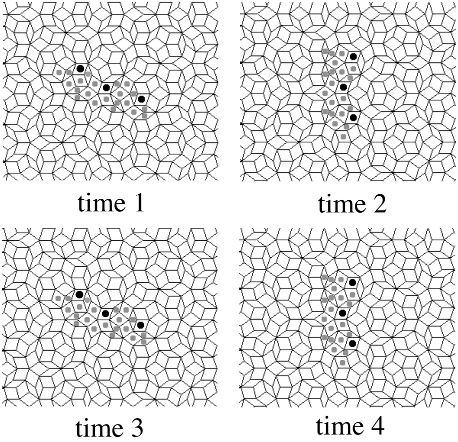
Redefine the notion of "neighbors" – define tiles to be neighbors if their preimages under the projection are neighbors in $\mathbb{Z}^5$.

Each cube projects to > 1 tile; pick a pair of axes in 5-space and identify a 5-d cube with the tile that is the image of that face in the Penrose tiling.

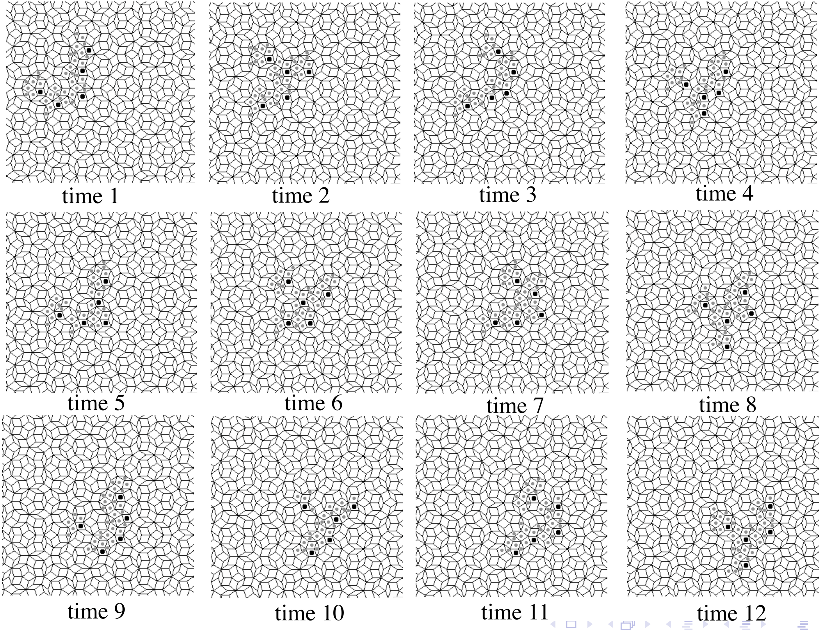
neighbors



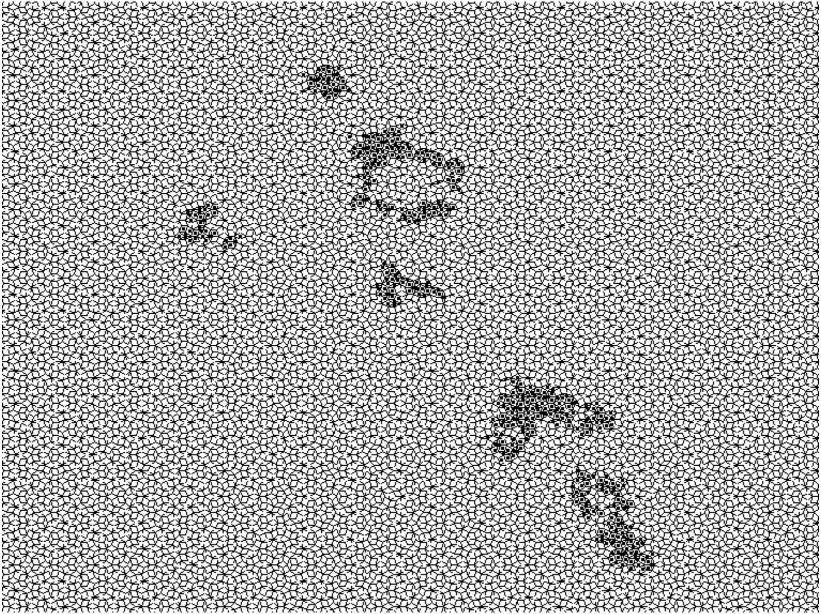
A small oscillator



A glider


time 1
time 2
time 3
time 4
time 5
time 6
time 7
time 8
time 9
time 10
time 11
time 12

Gosper glider gun



Movies!!!