18.312: Algebraic Combinatorics — Lionel Levine

Practice Problems for the Final

The questions on the exam on Thursday May 5 will be similar to these, but there will be fewer of them. Explain your reasoning to receive full credit, even for computational questions. If a result was proved in lecture or on the problem sets, you may use it without reproducing the proof.

PF1 Consider the poset $P = \{\widehat{0}, x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n, \widehat{1}\}$ with covering relations

$$\widehat{0} < x_i < y_j < z_k < \widehat{1}$$

for all $i, j, k \in [n]$. Find for each $\ell \geq 1$ the number of chains of length ℓ from $\widehat{0}$ to $\widehat{1}$, and use this information to determine $\mu_P(\widehat{0}, \widehat{1})$.

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PF2 Suppose P is a poset with $\widehat{0}$ and $\widehat{1}$ having $3\left(\binom{\ell+2}{3}\right) - 2\ell$ multichains $\widehat{0} \leq x_0 \leq x_1 \leq \ldots \leq x_\ell = \widehat{1}$ for each $\ell \geq 1$.

(a) What is the rank of P?

(b) What is $\mu_P(\widehat{0}, \widehat{1})$?

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PF3 Consider the 3-dimensional hyperplane arrangement

$$\mathcal{A} = \{x = y,\ x = -y,\ y = z,\ y = -z\}.$$

(a) Draw the Hasse diagram of the intersection lattice L of $\mathcal{A}$.

(b) Find $\mu(X, \widehat{1})$ for all $X \in L$.

(c) How many points $(x, y, z) \in \mathbb{F}_5^3$ belong to the union of the hyperplanes in $\mathcal{A}$?

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PF4 Let L be the lattice of linear subspaces of $\mathbb{F}_2^3$, ordered by inclusion.

(a) Draw the Hasse diagram of L.

(b) Find $\mu_L(0, 1)$.

(c) Find the number of linear maps $A : \mathbb{F}_2^3 \to \mathbb{F}_2^3$ such that $Av \neq v$ for all $v \in \mathbb{F}_2^3$.

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PF5 Let $A_1, \ldots, A_n, B_1, \ldots, B_n$ be sets of size 2 with $A_1 \cup \ldots \cup A_n = B_1 \cup \ldots \cup B_n = [2n]$. Prove that there is a permutation $\sigma \in S_n$ such that $A_{\sigma i} \cap B_i \neq \emptyset$ for all $i \in [n]$.

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PF6 Let G be a finite directed graph, and let $\kappa(G, v)$ be the number of oriented spanning trees of G rooted at v. Prove that if G is balanced (indeg(v)=outdeg(v) for all vertices v), then $\kappa(G, v) = \kappa(G, w)$ for all vertices v and w.

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PF7 Let $\ell, n \geq 1$. Find the number of closed paths of length ℓ in the complete graph K_n.

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PF8 Let G be a 3-regular undirected graph on 10 vertices whose adjacency matrix has eigenvalues $-2, -2, -2, -2, 1, 1, 1, 1, 1, 3$.

(a) Find the number of closed paths in G of length ℓ.

(b) How many spanning trees does G have?

(c) How many bi-Eulerian tours does G have, up to cyclic equivalence? (A bi-Eulerian tour is a closed path using every edge twice: once in each direction.)

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PF9 How many sequences $(x_1, \ldots, x_{91})$ with each $x_k \in [10]$ have the property that $x_1 = x_{91}$ and for every pair of distinct integers $i, j \in [10]$ there is exactly one $k \in [90]$ such that $(x_k, x_{k+1}) = (i, j)$?

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PF10 Let Y be the set of 3×3 matrices with entries in $[n]$. For $A, B \in Y$ define an equivalence relation $A \sim B$ if B can be obtained by permuting the rows and columns of A. Find the number of equivalence classes.