18.312: Algebraic Combinatorics Lionel Levine

Lecture 11

Lecture date: March 15, 2011 *Notes by: Ben Bond*

Today: Mobius Algebras, $\mu(\prod_n)$.

Test: The average was 17. If you got < 15, you have the option to hand in up to 3 problems from the practice midterm, (problems 1,7, and 11) for one point each. This will raise your score, but no higher than 15. This is due by Tuesday March 29.

1 Simplicial Complexes

Recall last time we proved a formula for the Mobiüs function for a finite poset P. Let $\hat{P} = 1 \oplus P \oplus 1 = P \cup \{\hat{0}, \hat{1}\}$. We showed

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = -c_1 + c_2 \dots (-1)^r c_r$$

where $c_i = \#\{\hat{0} = x_0 < x_1 < \ldots < x_i = \hat{1}\}$, i.e. the number of strict chains in $\hat{P}$ with minimal element $\hat{0}$, maximal element $\hat{1}$.

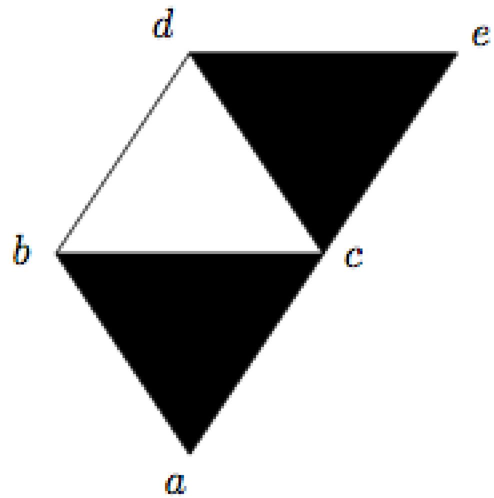
There is something topological in disguise here, we make the following definition:

Definition 1 *A* simplicial complex Δ *on a finite set* V *is a collection of subsets* $\Delta \subseteq 2^V$ *(*2^V *is the power set of* V*), satisfying:*

1. $\{x\} \in \Delta$ *for all* $x \in V$,
2. *If* $F \in \Delta$, *and* $G \subseteq F$, *then* $G \in \Delta$.

Remark 2 2^V *is a Boolean algebra, and* Δ *is an order ideal that contains all sets* $\{x\}$ *You should think of* Δ *as a set of (generalized) triangles glued together, as seen in the next example.*

Example 3 *Let* $V = \{a, b, c, d, e\}$, *and* $\Delta = \{a, b, c, d, e, ab, ac, bc, bd, cd, ce, de, abc, cde\}$ *(here* abc *denotes the set* $\{a, b, c\}$*) Think of each set as a simplex of dimension one less than the cardinality, i.e.* Δ *corresponds to the diagram below:*



The simplicial complex Δ *of Example 3*

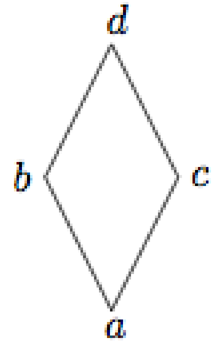
The sets in Δ are described by faces in the diagram. The triangles *abc*, *cde* are shaded because the sets $\{a, b, c\}, \{c, d, e\} \in \Delta$, but the set $\{b, d, e\} \notin \Delta$, so it does not appear as a shaded triangle in the diagram. The two-element sets correspond to lines, and one element sets correspond to points. If there had been a 4 element set, it would be drawn as a tetrahedron, etc.

In topology, we might have some complicated manifold, but by triangulating it, we get a simplicial complex, and may use combinatorics to better describe it.

To relate simplicial complexes to posets, we make the following definition:

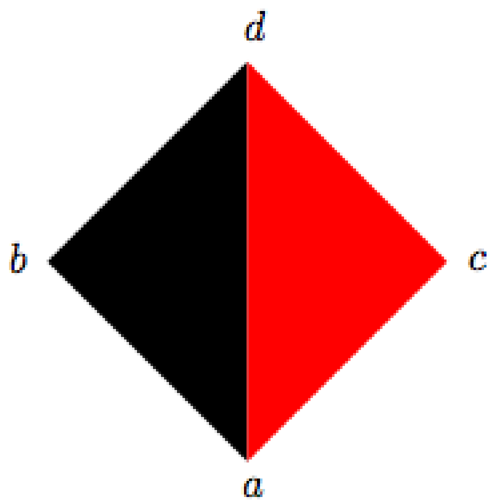
Definition 4 *Given a poset* P*, the* order complex *of* P *is the simplicial complex* $\Delta(P) = \{F \subseteq P \mid F \text{ totally ordered}\}$ *i.e.* $\Delta(P)$ *is all chains in* P*.*

Example 5 *Let* $P = B_2$*.* P *has Hasse diagram:*



The Hasse diagram of B_2

We see that the maximal chains are $a < b < d$ and $a < c < d$, so $\Delta(P)$ contains the sets $\{a, b, d\}$ and $\{a, c, d\}$. The simplicial complex is drawn below. (the two triangles have been colored differently to emphasize that there are two distinct triangles)



The Order Complex of B_2

Definition 6 *The elements $F \in \Delta$ are called* faces*. The dimension of a face is defined as:*

$$\dim F := |F| - 1$$

Remark 7 *The definition of dimension corresponds with what we would expect geometrically. For example, a triangle is defined by its three vertices, and has dimension 2.*

We now introduce the simplest topological invariant:

Definition 8 *The* Euler Characteristic *of a simplicial complex* Δ *is:*

$$\chi(\Delta) = f_0 - f_1 + \ldots + (-1)^d f_d$$

Where f_i *is the number of faces of dimension* i*, and* $d = \dim \Delta := \max\{\dim_{F \in \Delta} F\}$.

Topologically equivalent simplicial complexes give the same value of $\chi(\Delta)$. Notice that the formula for $\chi(\Delta)$ is similar to $\mu_{\hat{P}}$, seen above. In fact,

Proposition 9 *We have*

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = \chi(\Delta(P)) - 1$$

Remark 10 $\chi(\Delta(P)) - 1$ *is known as the reduced Euler characteristic.*

Proof: Notice that c_k is the number of chains with $k+1$ elements. Since the minimal and maximal elements are $\hat{0}$ and $\hat{1}$, this corresponds to finding elements of $\Delta(P)$ with $k-1$ elements, i.e. dimension $k-2$. The number of these is the coefficient f_{k-2}. Thus we have a correspondence between the f values and c values, except for there is no f value corresponding to c_1. Since $c_1 = 1$, we must subtract 1 from $\chi(\Delta(P))$ to make the two equal. □

2 Mobiiüs Algebras

Definition 11 *Let* L *be a lattice,* K *a field. The* Mobiiüs Algebra $A(L)$ *is defined as,*

$$A(L) = \{\textit{formal sums} \sum_{x \in L} a_x x \mid a_x \in K\}$$

with multiplication $x \cdot y = x \wedge y$

Notice that unlike the incidence algebra, the Mobiiüs Algebra is commutative.

Definition 12 *We define:*

$$\delta_x = \sum_{y \leq x} \mu(y, x) x$$

By Mobiüs inversion, we have

$$x = \sum_{y \leq x} \delta_y$$

Since $\{x\}_{x \in L}$ form a basis for $A(L)$, and each x may be written as a linear combination of elements δ_x, we see that $\{\delta_x\}_{x \in L}$, span $A(L)$. Since $\#\{\delta_x\}_{x \in L} = \#\{x\}_{x \in L}$, this means $\{\delta_x\}_{x \in L}$ form a basis of $A(L)$.

We now take a quick aside to define direct sums of K-algebras.

Definition 13 *Let A, B be K-algebras. Their direct sum is,*

$$A \oplus B = \{\textit{formal sums } a + b \mid a \in A, b \in B\}$$

Multiplication is given by $(a + b)(a' + b') = aa' + bb'$

Remark 14 *Notice $A, B \subset A \oplus B$, by letting a or b be 0 in the definition of $A \oplus B$. Also, multiplication can be thought of as letting multiplication from A and B carry over to $A \oplus B$, and defining $ab = 0$ for $a \in A, b \in B$.*

We now define a map from $A(L)$ to a direct sum of copies of the field K.

Definition 15 *In the space $\bigoplus_{x \in L} K$, let e_x denote the identity element of the field in the sum corresponding to index x. Then we define*

$$\theta : A(L) \to \bigoplus_{x \in L} K$$

such that $\theta(\delta_x) = e_x$.

The best way to think about $\bigoplus_{x \in L} K$ is as a vector space with basis $\{e_x\}_{x \in L}$. An arbitrary element is $\sum_{x \in L} c_x e_x$, for $c_x \in K$. As an algebra, we have multiplication of basis elements $e_x e_y = 0$ for $x \neq y$, and $e_x e_x = e_x$. Multiplication of general vectors follows from this definition, for $c_x, d_x \in K$,

$$\left(\sum_{x \in L} c_x e_x\right)\left(\sum_{x \in L} d_x e_x\right) = \sum_{x \in L} c_x d_x e_x.$$

Proposition 16 *The map θ is an isomporphism of K algebras.*

Proof: Since $\{\delta_x\}_{x\in L}$ form a basis of $A(L)$, and $\{e_x\}_{x\in L}$ form a basis of $\bigoplus_{x\in L} K$, and θ is a linear map giving a bijective correspondence between basis elements, we see θ is an isomorphism of vector spaces. To extend this to an isomorphism of algebras, we need to check that $\theta(xy) = \theta(x)\theta(y)$ for $x, y \in A(L)$ (although θ is defined in terms of δ_x, it is easiest to check multiplication in the basis $\{x\}_{x\in L}$). Using the formula for x given after Definition 12, we get:

$$\theta(xy) = \theta(x \wedge y) = \theta\left(\sum_{z\leq x\wedge y} \delta_z\right) = \sum_{z\leq x\wedge y} \theta(\delta_z) = \sum_{z\leq x\wedge y} e_z$$

The last two equalities come from linearity of θ and the definition of θ. We now evalutate $\theta(x)\theta(y)$. This gives:

$$\theta(x)\theta(y) = \left(\sum_{u\leq x} e_u\right)\left(\sum_{w\leq y} e_w\right) = \left(\sum_{\substack{u\leq x\\ w\leq w}} e_u e_w\right)$$

Notice $e_u e_w = 0$ unless $u = w$. This means the only remaining terms are e_u for $u \leq x, y$, i.e., $u \leq x \wedge y$, which gives

$$\theta(x)\theta(y) = \sum_{u\leq x\wedge y} e_u$$

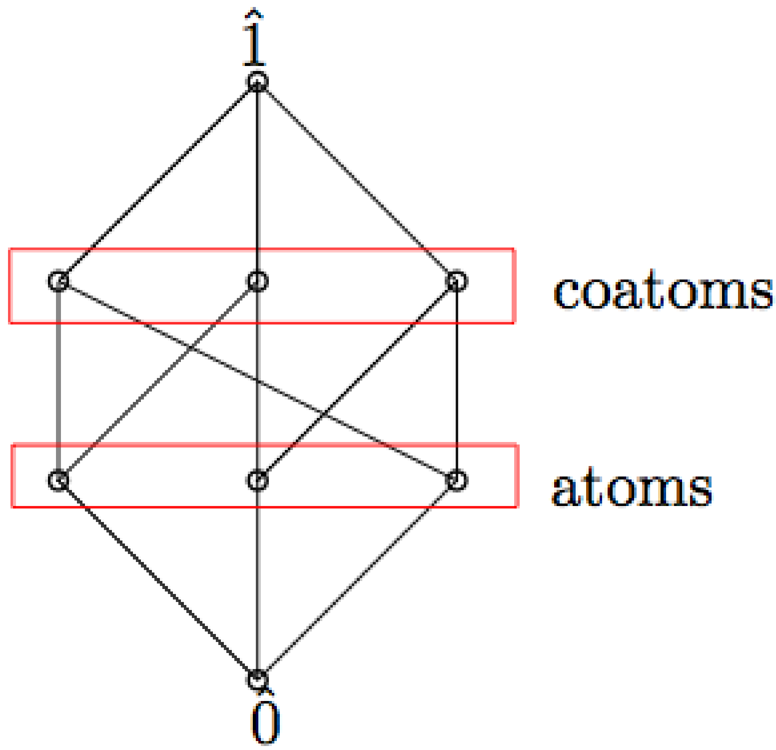
This is equal to $\theta(xy)$ as calculated above, thus $\theta(xy) = \theta(x)\theta(y)$, and θ is an isomorphism of algebras. $\square$

This may seem a little disappointing. We would hope that the Mobiüs algebra would describe the combinatorial properties of the lattice, but from Proposition 16 we see that any two lattices with the same cardinality have isomorphic Mobiüs algebras. However, the Mobiüs algebra has some useful applications as wee will see shortly.

3 Atoms and Coatoms

Definition 17 *An* atom *of a lattice L is a minimal element of $L - \{\hat{0}\}$. Dually, a* coatom *is a maximal element of $L - \{\hat{1}\}$.*

Example 18 *In $L = B_3$, we have:*



Lemma 19 *Let $X \subset L$ be a subset satisfying*

1. $\hat{1} \notin X$
2. *X contains all coatoms of L*

Then

$$\mu_L(\hat{0}, \hat{1}) = \sum_{k \geq 1} (-1)^k N_k$$

where $N_k = \#\{S \subset X \mid \#S = k$ and $\bigwedge_{y \in S} y = \hat{0}\}$.

Proof: Consider $\prod_{x \in X}(\hat{1} - x)$. By expanding the sum (notice that $\hat{1}$ is the identity element, $\hat{1} \wedge x = x$), we get:

$$\prod_{x \in X}(\hat{1} - x) = \sum_{S \subseteq X} (-1)^{\#S} \bigwedge_{y \in S} y$$

Write this in terms of the basis $\{x\}_{x \in L}$ to get

$$\sum_{S \subseteq X} (-1)^{\#S} \bigwedge_{y \in S} y = \sum_{x \in L} c_x x$$

When we evaluate the coefficient $c_{\hat{0}}$ of $\hat{0}$, we find,

$$c_{\hat{0}} = \sum_{\substack{S \subseteq X \\ \bigwedge_{y \in S = \hat{0}}}} (-1)^{\# S}$$

Applying the definition of N_k given in the statement of the theorem, we find $c_{\hat{0}} = \sum_{k \geq 1} (-1)^k N_k$. Notice that k cannot be 0, because the meet over the empty set is $\hat{1}$ by definition.

We now evaluate $\prod_{x \in X} (\hat{1} - x)$ in a different way to obtain an alternate expression for $c_{\hat{0}}$. Using the relation between the bases $\{x\}$ and $\{\delta_x\}$, we may write:

$$\hat{1} - x = \sum_{y \leq \hat{1}} \delta_y - \sum_{y \leq x} \delta_y = \sum_{y \not\leq x} \delta_y. \tag{1}$$

Notice that by the isomorphism θ with $\bigoplus_{x \in L} K$, $\delta_y \delta_z = 0$ for $y \neq z$, and $\delta_y \delta_y = \delta_y$. Thus when we substitute equation (1) into $\prod_{x \in X} (\hat{1} - x)$ we get:

$$\prod_{x \in X} (\hat{1} - x) = \sum_{y \in Y} \delta_y$$

where $Y = \{y \in L | y \not\leq x, \forall x \in X\}$. Notice that X contains all coatoms by assumption, and the only element of L greater than all coatoms is $\hat{1}$. Since $\hat{1} \notin X$, $\hat{1}$ is the only element of Y. Thus $\prod_{x \in X} (\hat{1} - x) = \delta_{\hat{1}}$. Recall that $\delta_{\hat{1}} = \sum_{y \leq \hat{1}} \mu_L(y, \hat{1}) y$. By comparing the coefficient of $c_{\hat{0}}$ for this description of $\prod_{x \in X} (\hat{1} - x)$, and the one that was obtained earlier, we find that

$$\mu_L(\hat{0}, \hat{1}) = \sum_{k \geq 1} (-1)^k N_k,$$

as desired. □

The use of this formula is not immediately clear, however, we present the following corollary:

Corollary 20 *If $\hat{0}$ is not a meet of coatoms of L, then $\mu_L(\hat{0}, \hat{1}) = 0$. Dually, if $\hat{1}$ is not a join of atoms, then $\mu_L(\hat{0}, \hat{1}) = 0$.*

Example 21 *A simple example of such a lattice is a chain. There is only one atom and one coatom, so $\hat{1}$ is not a join of atoms, and $\hat{0}$ is not the meet of coatoms.*

Proof: To prove (1), let $X = \{\text{coatoms}\}$, and apply Lemma 19. Then $N_k = 0$ for all k. This follows from the definition of N_k and the assumption that $\hat{0}$ is not a meet of coatoms. Thus by Lemma 19, $\mu_L(\hat{0}, \hat{1}) = 0$.

To prove (2), notice that the atoms of L are in bijective correspondence with the coatoms of L^*, and that $\zeta_L(x,y) = \zeta_{L^*}(y,x)$, i.e. $\zeta_{L^*} = (\zeta_L)^T$. Since the operations transpose and matrix inverse commute, we find:

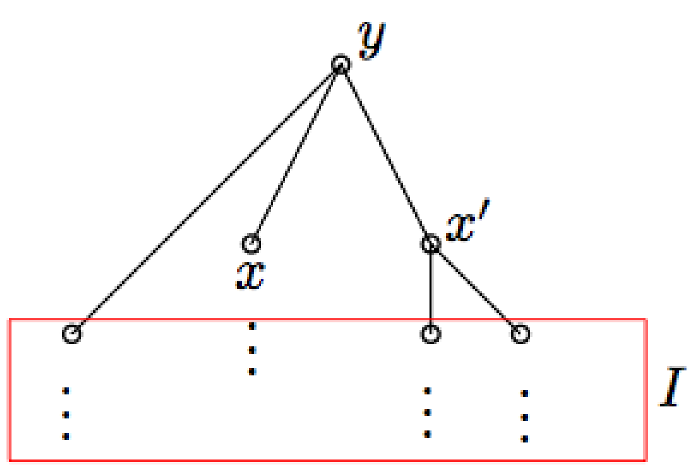
$$\mu_{L^*} = (\zeta_{L^*})^{-1} = ((\zeta_L)^T)^{-1} = (\zeta_L^{-1})^T = \mu_L^T$$

Thus $\mu_L(\hat{0},\hat{1}) = \mu_{L^*}(\hat{1},\hat{0})$, so by part (1), $\mu_{L^*}(\hat{1},\hat{0}) = 0$ □

Example 22

Let L be a finite distributive lattice. By Birkhoff's theorem, $L = J(P)$ for some poset P. We wish to compute μ_L. Notice that each lattice for which we have found μ, it is distributive, so all our previous examples will be a corollary of this example. Let $[I,I']$ be an interval in $J(P)$, i.e. $I \subseteq I'$ are order ideals of P. We compute $\mu_L(I,I')$. Notice that $[I,I']$ is an interval of a distributive lattice, hence a distributive lattice, so $[I,I'] = J(Q)$ for some Q. We claim that $Q = I' - I$. To see this notice we may make a map from $[I,I']$ to $J(I'-I)$ by sending an element $I'' \in [I,I']$ to $I'' - I \in J(Q)$. We can see that $I'' - I$ is indeed an order ideal of Q, because it is closed under going down the lattice. To see the inverse map is well defined, say $H \subset Q$ is an order ideal, so $H \cup I$ is also an order ideal (of P) containing I, hence $H \cup I \in [I,I']$.

Now look at atoms of $[I,I']$. They must all be of the form $I \cup \{x\}$, where x covers an element of I. For example, in the Hasse diagram of P below, we see that $I \cup \{x\}$ and $I \cup \{x'\}$ both cover I (which is the $\hat{0}$ element of $[I,I']$), so they are atoms). However, $I \cup \{y\}$ is not an order ideal, so is not an atom. Any order ideal containing y must also contain x and x', hence cannot be an atom.



Example of a Hasse diagram of P

From this we see that the join of all atoms is $I \cup M$, where $M = \{x \in P \mid x \text{ covers some element of } I\}$. When is the maximal element I' a join of atoms? When $I' = I \cup M$. For example, this is not true in the above diagram, because $I' = I \cup \{x, x', y\}$ is not a join of atoms.

We claim that $I' = I \cup M$ if and only if Q is an antichain. To see this, notice that if any two elements were comparable, then the larger would not be contained in an ideal that is a join of atoms.

In the case that Q is an antichain, we get $[I, I'] = J(Q) = B_n$, where $n = |Q| = |I'| - |I|$. This means $[I, I']$ is a boolean algebra, so we may explicitly write down $\mu(I, I')$ (it was computed in lecture 10). Thus we have proved the following theorem:

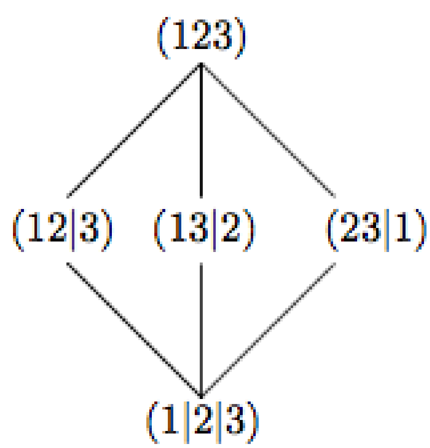
Theorem 23 *Let $L = J(P)$ be a finite distributive lattice, and $I \subset I' \in L$. Then:*

$$\mu_L(I, I') = \begin{cases} 0 & [I, I'] \ncong B_n \\ (-1)^n & [I, I'] \cong B_n \end{cases}$$

4 Computing $\mu(\prod_n)$

Recall $\prod_n$ is the poset of partitions of n, ordered by refinement. We would like to compute $\mu_{\prod_n}(\sigma, \tau)$.

Example 24 *We first draw the Hasse diagram of $\prod_3$. The notation $(12|3)$ means a partition into a block contain 1,2, and a block containing 3.*



Hasse Diagram of $\prod_3$

Notice that $(12|3) \vee ((13|2) \wedge (23|1)) = (12|3) \vee (1|2|3|) = (12|3)$ *and that* $((12|3) \vee (13|2) \wedge ((12|3) \vee (23|1)) = (123) \wedge (123) = (123)$, *so this lattice is not distributive. Thus we cannot apply Theorem 23.*

To calculate $\mu_{\prod_n}(\sigma, \tau)$, notice that $\sigma \leq \tau$ means every block of σ is contained in a block of τ, and every block of τ is a disjoint union of blocks of σ. Write $\tau = (\tau_1, \ldots, \tau_k)$, and say that $\tau_i = \bigcup_{j \in B_i} \sigma_j$. The B_i index which σ_j the block τ_i is composed of. Let $\lambda_i = |B_i|$. If $\pi \in [\sigma, \tau]$, then π satisfies:

1. $\sigma \leq \pi$, so each block of π is a union of blocks of σ.
2. $\pi \leq \tau$, so each block of π is contained in a block of τ.

We claim that:

$$[\sigma, \tau] \cong \prod_{\lambda_1} \times \prod_{\lambda_2} \times \ldots \times \prod_{\lambda_k}$$

Proof: Choosing $\pi \in [\sigma, \tau]$ is equivalent to choosing a partition of B_i for $i = 1 \ldots k$. Since $|B_i| = \lambda_i$, this is just a partition of λ_i. $\square$

Using the formula for μ of a product poset, we find:

$$\mu_{\prod_n}(\sigma, \tau) = m_{\lambda_1} m_{\lambda_2} \ldots m_{\lambda_k}$$

where $m_l = \mu(\prod_l)$. This is new notation, defined by $\mu(\prod_l) = \mu_{\prod_l}(\hat{0}, \hat{1})$. It is motivated by the fact that each interval in a lattice is a lattice, so we may think of μ as a function on lattices.

This example will be finished next class.