18.312: Algebraic Combinatorics	Lionel Levine
Lecture 12	
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This lecture:

- A continuation of the last lecture: computation of μ_{Π_n}, the Möbius function over the incidence algebra of partition lattices.
- The zeta polynomial of the poset P.
- Finite Boolean algebras.
- A review of selected questions from the midterm.

1 Computing μ_{Π_n}, continued...

In lecture 11 we discussed partition lattices, and showed that intervals $[\sigma, \tau]$ of the lattice were isomorphic to a direct product of k posets, where k is the number of blocks of τ. In particular, given an interval $[\sigma, \tau]$ of Π_n, if τ has k blocks, each the (disjoint) union of λ_k blocks of σ, then

$$[\sigma, \tau] \simeq \Pi_{\lambda_1} \times \cdots \times \Pi_{\lambda_k}$$

Using our earlier lemma for the Möbius function of a direct product we can write

$$\mu_{\Pi_n}[\sigma, \tau] = \mu_{\Pi_{\lambda_1}} \times \cdots \times \mu_{\Pi_{\lambda_k}} \tag{1}$$

where

$$\mu_{\Pi_\lambda} \equiv \mu_{\Pi_\lambda}\left(\hat{0}, \hat{1}\right)$$

Lemma 1 *(lattice recurrence) Let L be a lattice with $|L| \geq 2$, and recalling that a co-atom is a maximal element of $L - \left\{\hat{1}\right\}$, fix a co-atom $a \in L$. Then*

$$\sum_{\substack{x \in L \\ x \wedge a = \hat{0}}} \mu\left(x, \hat{1}\right) = 0$$

Proof:

We use the following facts about the Möbius algebra $A(L)$ which we discussed in lecture 11:

$$x = \sum_{y \leq x} \delta_y \tag{2}$$

$$\delta_x = \sum_{y \leq x} \mu(y, x)\, y \tag{3}$$

$$\delta_x \delta_y = \begin{cases} \delta_x & , x = y \\ 0 & , \text{otherwise} \end{cases} \tag{4}$$

From equations (2) and (4) and the assumption that the co-atom $a \neq \hat{1}$

$$a\delta_{\hat{1}} = \left(\sum_{y \leq a} \delta_y\right) \delta_{\hat{1}} = 0 \Rightarrow a\delta_{\hat{1}} = \sum_{x \in L} c_x x = 0$$

and since $a\delta_{\hat{1}}$ is identically zero, all coefficients of $a\delta_{\hat{1}}$ in the natural basis of $A(L)$ are zero, in particular the coefficient $c_{\hat{0}}$ of $\hat{0}$ vanishes. From the multiplication rule for $A(L)$ and using equation (3) applied to $\delta_{\hat{1}}$ we can also write

$$0 = a\delta_{\hat{1}} = a \sum_{y \leq \hat{1}} \mu\left(y, \hat{1}\right) y = \sum_{y \leq \hat{1}} \mu\left(y, \hat{1}\right) (a \wedge y) \tag{5}$$

so if we restrict the sum in equation (5) to $y \in L$ such that $(a \wedge y) = \hat{0}$, then we can equate the sum of the Möbius functions with the coefficient $c_{\hat{0}}$, which is identically zero.

□

Applying Lemma 1 to $L = \Pi_n$, we can pick co-atoms a_i with partitions whose two blocks are $\{i\}$ and $[n] - \{i\}$. The lemma condition $x \wedge a_i = \hat{0}$ implies that either $x = \hat{0}$ or x has a total of $n-1$ blocks, $n-2$ blocks that are singletons, and one block of two elements, one of which is i. Denote partitions of this sort by x_i. The lemma then states that for each co-atom $a \in \{a_i\}_{i=1}^n$

$$\sum_{\substack{x \in L \\ x \wedge a = \hat{0}}} \mu_{\Pi_n}\left(x, \hat{1}\right) = \mu_{\Pi_n}\left(\hat{0}, \hat{1}\right) + \sum_{i=1}^{n-1} \mu_{\Pi_n}\left(x_i, \hat{1}\right) = 0$$

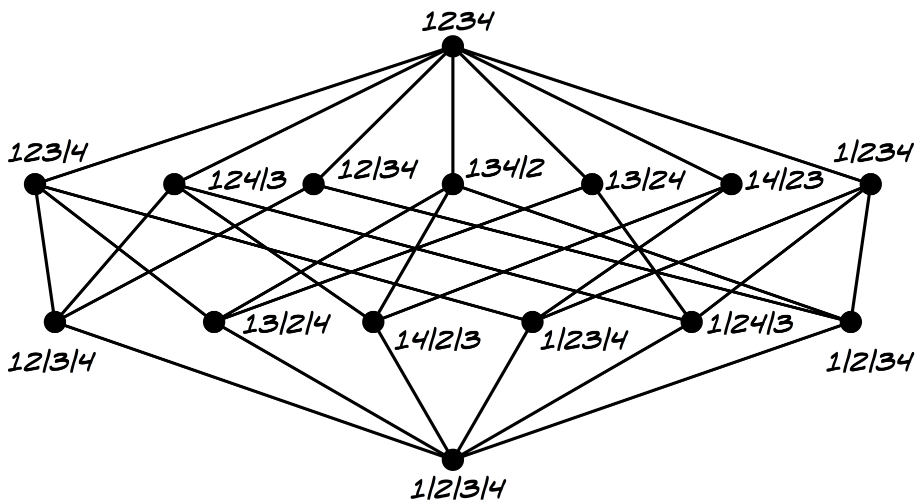
which after re-arranging and using the fact that $\left[x_i, \hat{1}\right] \simeq \Pi_{n-1}$ (so $\mu_{\Pi_n}\left(x_i, \hat{1}\right) = \mu_{\Pi_{n-1}}\left(\hat{0}, \hat{1}\right) \equiv \mu_{\Pi_{n-1}}$) gives

$$\begin{aligned} \mu_{\Pi_n}\left(\hat{0}, \hat{1}\right) &= -\sum_{i=1}^{n-1} \mu_{\Pi_n}\left(x_i, \hat{1}\right) \\ &= -(n-1)\, \mu_{\Pi_{n-1}} \\ &= (-1)^{n-1} (n-1)! \end{aligned}$$

and so, using equation (1), we see that in general

$$\begin{aligned} \mu_{\Pi_n}(\sigma,\tau) &= \mu_{\Pi_{\lambda_1}} \times \cdots \times \mu_{\Pi_{\lambda_k}} \\ &= \prod_{i\in[k]} (-1)^{\lambda_i - 1} (\lambda_i - 1)! \end{aligned}$$

Example 2 *The Hasse diagram of* Π_4 *is shown below:*



HASSE DIAGRAM OF Π_4

and the corresponding matrix of Möbius function values is

$$\begin{array}{c} 1|2|3|4 \\ 12|3|4 \\ 13|2|4 \\ 14|2|3 \\ 1|23|4 \\ 1|24|3 \\ 1|2|34 \\ 123|4 \\ 124|3 \\ 12|34 \\ 134|2 \\ 13|24 \\ 14|23 \\ 1|234 \\ 1234 \end{array} \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & -6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where the rows of the matrix are labeled with the partition represented by the corresponding vertex of the Hasse diagram.

We have $\hat{0} = 1|2|3|4$, *i.e. the partition with four blocks, and the values of* $\mu\left(\hat{0},\cdot\right)$ *are in row* 1 *of the matrix above, while* $\hat{1} = 1234$, *the partition having a single block (row* 15 *in* *the matrix). The co-atoms each have two blocks, and their Möbius function values are in rows* 8 *though* 14*, and so per Lemma 1, if we choose the co-atom* $a = 123|4$ *(with* $\mu(a, \cdot)$ *values in row* 8*), then the elements of* $x \in \Pi_4$ *such that* $x \wedge a = \hat{0}$ *are the vertices labeled* $14|2|3, 1|24|3, 1|2|34$ *(rows* $4, 6, 7$ *respectively). From the Möbius function we can confirm*

$$\begin{aligned} -6 = \mu\left(\hat{0}, \hat{1}\right) &= -\left(\mu\left(4, \hat{1}\right) + \mu\left(6, \hat{1}\right) + \mu\left(7, \hat{1}\right)\right) \\ &= -\left(2 + 2 + 2\right) \end{aligned}$$

Similarly, if we choose co-atom $a = 12|34$ *(row* 10*) then the elements of* $x \in \Pi_4$ *such that* $x \wedge a = \hat{0}$ *correspond to rows* $3, 4, 5, 6, 12, 13$*. From the Möbius function values in those rows of the matrix we can confirm*

$$-6 = \mu\left(\hat{0}, \hat{1}\right) = \sum_{x \in \{3,4,5,6,12,13\}} \mu\left(x, \hat{1}\right)$$

2 Zeta polynomial of a poset.

For a poset P with minimum and maximum elements $\hat{0}, \hat{1}$ respectively, define a function of n as follows

$$Z(P, n) \equiv \#\left\{ \text{multichains: } \hat{0} = x_0 \leq x_1 \leq \cdots \leq x_n = \hat{1} \,\middle|\, x_i \in P \right\} \tag{6}$$

then using the ideas we developed for incidence algebras we can write this polynomial in n in terms of the zeta function on P, i.e. as $\zeta^n\left(\hat{0}, \hat{1}\right)$. By contrast with (6), the zeta functions is well formed for all $n \in \mathbb{Z}$.

Claim: $Z(P, n)$ is a polynomial in n and can be shown to satisfy a linear recurrence.

Recall that $(\zeta - 1)^{r+1} = 0$ if P is of rank r, so let

$$Z_n \equiv Z(P, n)$$

then Z_n satisfies the recurrence

$$(E - 1)^{r+1} Z = 0$$

which implies that Z_n is a polynomial $q(\cdot)$ in n of degree $\leq r$, and in fact, the degree of $q(\cdot)$ is equal to r.

This gives another way to compute Möbius functions, since $Z(P, -1) = \zeta^{-1}\left(\hat{0}, \hat{1}\right) = \mu\left(\hat{0}, \hat{1}\right)$.

Example 3 *(zeta polynomial on the Boolean algebra of rank* r*) Let* $P = B_r$*, then the zeta polynomial* $Z(B_r, n)$ *counts multi-chains of the form*

$$\tilde{A}_{\text{¿}} = \hat{0} \subseteq S_0 \subseteq \cdots \subseteq S_n = \hat{1} = [r]$$

and by construction, each $i \in [r]$ appears for the first time in some S_j, $j \in [n]$, which we can choose independently. Thus the number of multi chains is n^r, since there are n choices for the set where $i \in [r]$ appears for the first time, and there are r elements of $[r]$. Thus

$$Z\left(B_r, -1\right) = \mu_{B_r}\left(\hat{0}, \hat{1}\right) = (-1)^r$$

3 Lattice Axioms.

We assert the following lattice axioms

$x \vee y = x \vee y$	$x \wedge y = x \wedge y$
$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$
$x \wedge (x \vee y) = x$	$x \vee (x \wedge y) = x$

where the axioms of the last line are referred to as *absorption* axioms. With the identification

$$x \leq y \equiv x = x \wedge y$$

we can check the following lattice properties:

1. does $x \leq x \Rightarrow x = x \wedge x$?

$$\begin{aligned} x &= x \vee (x \wedge x) \text{ by absorption} \\ x \wedge (x \vee (x \wedge x)) &= x \wedge x \text{ by absorption again} \\ &= x \end{aligned}$$

2. do $x \leq y$, and $y \leq x \Rightarrow x = y$?

$$\begin{aligned} x \leq y &\equiv x = x \wedge y \\ y \leq x &\equiv y \wedge x = y \\ &\Rightarrow x = y \end{aligned}$$

3. does transitivity hold?

$$\begin{aligned} x \leq y \,\&\, y \leq z &\Rightarrow x = x \wedge y \,\&\, y = y \wedge z \\ &\Rightarrow x \wedge (y \wedge z) = (x \wedge y) \wedge z = x \wedge z \\ &\Rightarrow x \leq z \end{aligned}$$

So, the three axioms above satisfy the requirements for a poset lattice. If we add the axiom

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

we can also describe a distributive lattice axiomatically.

Finally, we can add the following complement axiom. Assume $\exists \hat{0}, \hat{1}$ and $\forall x$, $\exists \neg x$ such that $\hat{0} = x \wedge (\neg x)$ and $\hat{1} = x \vee (\neg x)$. Then we can describe the Boolean algebra as follows

Theorem 4 *If L is a finite Boolean algebra by axiomatic definition, then $L \simeq B_n$ for some $n \in \mathbb{N}$.*

Proof:

By Birkhoff's Theorem, since L is a distributive lattice, we have $L = J(P)$ for some poset P. If L is to be isomorphic to B_n then we need to show that P is an antichain, i.e. $P = \underline{1} + \underline{1} + \cdots + \underline{1}$ is the direct sum of n singletons, and has no order relations (there do not exist $x, y \in P$ such that $x < y$). To this end, assume the complement axiom holds so that for $I \in L = J(P)$

$$\begin{aligned} I \wedge (\neg I) = I \cap (\neg I) &= \emptyset = \hat{0} \\ I \vee (\neg I) = I \cap (\neg I) &= P = \hat{1} \end{aligned}$$

and so the complement operator must be set-theoretic complement in this instance:

$$\neg I = P - I.$$

However, if P has a nontrivial order relation $x < y$, then consider the principal ideal $I = \langle x \rangle = \{ z \in P | \, z \leq x \}$. Its complement $P - I$ is not an order ideal, since $y \in P - I$ and $x < y$ but $x \notin (P - I)$. Therefore L does not satisfy the complement axioms unless P is an antichain. $\square$

In logic, if L is a Boolean algebra, the elements of L can be interpreted as *propositions* or *sentences* with

$$\begin{aligned} x \wedge y &\equiv x \text{ and } y \\ x \vee y &\equiv x \text{ or } y \\ \neg x &\equiv \text{not } x \\ \hat{0} &\equiv \text{FALSE} \\ \hat{1} &\equiv \text{TRUE} \end{aligned}$$

for $x, y \in L$.

Example 5 $B_1 \simeq \underline{2} = \{\hat{0}, \hat{1}\}$

Example 6 $B_n \simeq \underline{2} \times \underline{2} \times \cdots \times \underline{2}$ *(n bits)*

4 Midterm review.

Question 7 *Midterm question M5.*

Answer 8 $a_{n+2} - 4a_{n+1} + 4a_n = 0 \Rightarrow \left(E^2 - 4E + 4\right) a_n = (E-2)^2 a_n = 0$, *which then means that*

$$\begin{aligned} a_n &= r2^n + ns2^n \\ r &= a_0 \\ s &= \frac{a_1}{2} - a_0 \end{aligned}$$

then

- *(c)*

$$\begin{aligned} b_n &= 2^{-n} a_n \\ &= r + ns \\ &= (r + ns)(1)^n \\ &\Rightarrow (E-1)^2 b_n = 0 \end{aligned}$$

- *(d)*

$$\begin{aligned} c_n &= a_n - 2 \\ &= r2^n + ns2^n - 2(1)^n \\ &\Rightarrow (E-2)^2 (E-1) c_n = 0 \end{aligned}$$

- *(e)*

$$\begin{aligned} d_n &= a_{2n} = r2^{2n} + 2ns2^{2n} \\ &= r4^n + 2ns4^n \\ &\Rightarrow (E-4)^2 d_n = 0 \end{aligned}$$

Question 9 *Midterm question M1.*

Answer 10 *We want to show that* $\binom{2p}{p} - \binom{2}{1}$ *is divisible by* p, *for* p *prime. Framed as a necklace problem, consider necklaces* $\underline{a}$ *composed of* $2p$ *beads with* p *red beads (say), and* p *blue beads. There are* $\binom{2p}{p}$ *necklaces that fit this description, and since* p *is prime, the possible stabilizers are* C_1, C_2, C_p, C_{2p}, *and*

$$\begin{array}{rcl} C_{2p} & - & \textit{no necklaces} \\ C_p & - & 2 \textit{ necklaces with alternative colors} \\ C_2 & - & \textit{no necklaces except for } p = 2 \\ C_1 & - & \left(\binom{2p}{p} - 2\right) \textit{ necklaces} \end{array}$$

the last number is divisible by $2p$, *and so also by* p.

Answer 11 *Writing the Vandermonde convolution*

$$\begin{aligned} \sum_{k=0}^{p} \binom{p}{k}\binom{p}{p-k} &= \binom{2p}{p} \\ 2 + \sum_{k=1}^{p-1} \binom{p}{k}\binom{p}{p-k} &= \binom{2p}{p} \\ &\Rightarrow \binom{2p}{p} \textit{ mod } p = 2 \end{aligned}$$

since

$$\binom{p}{k} = \frac{p!}{k!\,(p-k)!}$$

is divisible by p *for* $k \in [p]$. *Thus*

$$\binom{2p}{p} - 2 \textit{ mod } p = 0$$