18.312: Algebraic Combinatorics	Lionel Levine
Lecture 15	
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1 The Braid Arrangement

In the last lecture we were introduced to the characteristic polynomial, $\chi(\mathcal{A}, q)$, of the hyperplane arrangement $\mathcal{A} = \{H_1, ..., H_n\}$ which was defined as:

$$\chi(\mathcal{A}, q) = \#(\mathbb{F}_q^l - \cup_{i=1}^n H_i) = \sum_{X \in L(\mathcal{A})} \mu(X, \hat{1}) q^{dimX}.$$

We were then introduced to the Braid arrangement which we redefine here.

Definition 1 *The Braid arrangement, B_l is defined as:*

$$B_l = \{H_{ij}\}_{1 \leq i < j \leq l}.$$

Where:

$$H_{ij} = \{v \in \mathbb{F}_q^l | v_i = v_j\}.$$

We start our study of the Braid arrangment by studying the intersection poset (defined last lecture) of B_l. We find that:

$$L(B_l) = \hat{\Pi}_l = \text{lattice of partitions of } l,$$

With minimum element:

$$\cap H_{ij} = \{v \in \mathbb{F}_q^l | v_1 = v_2 = ... = v_l\}.$$

Next we wish to study the characteristic polynomial of B_l. To begin we can choose v_1 in q ways. We can then choose v_2 in $q - 1$ ways because it must be distinct from v_1. Similarly we can choose v_3 in $q - 2$ ways because it must be distinct from v_1 and v_2. Continuing this line of reasoning we find:

$$\chi(B_l, q) = q(q - 1)(q - 2)...(q - l + 1).$$

You may recall from our study of Stirling numbers that that this is the same equation appeared during our study signed stirling numbers of the first kind. Given the signed stirling numbers of the first kind, $s(l,k)$, we have:

$$\chi(B_l, q) = \sum_{k=1}^{l} s(l,k) q^k.$$

It is interesting that signed Stirling numbers of the first kind appear here, because we recall that it is signed Stirling numbers of the first kind that count partitions, and we have already seen a close connection between B_l and partitions. It turns out that they key to the relation here is Stirling reciprocity, which we have already seen in class, but for which we can now present a different proof. To begin we recall our definition of Stirling numbers of the second kind $S(k,j)$.

$$\begin{aligned} S(k,j) &= \#\{\text{partitions of [k] into j nonempty parts}\} \\ &= \#\{\lambda \in \Pi_k : \ |\lambda| = j.\} \end{aligned}$$

If we consider some fixed $\pi \in \Pi_l$ where $|\pi| = k$ we can then write:

$$S(k,j) = \#\{\lambda \in \Pi_l : |\lambda| = j, \lambda \geq \pi\}.$$

For signed Stirling numbers of the first kind we have seen:

$$\sum_{k=1}^{l} s(l,k) q^k = \chi(B_l, q) = \sum_{X \in L(B_n)} \mu(X, \hat{1}) q^{dimX}.$$

Which means:

$$\begin{aligned} s(l,k) &= \sum_{\substack{X \in L(B_n) \\ dimX = k}} \mu(X, \hat{1}) \\ &= \sum_{\substack{\pi \in \hat{\Pi}_l \\ |\pi| = k}} \mu(\hat{0}, \pi). \end{aligned}$$

We are now prepared to prove Stirling reciporcity.

Theorem 2 *If we have signed Stirling numbers of the first kind, s(l,k), and Stirling numbers of the second kind, S(k,j), then:*

$$\sum_{k=j}^{l} s(l,k) S(k,j) = \delta_{jl} = \begin{cases} 0 & if\ j \neq l \\ 1 & if\ j = l \end{cases}$$

This concludes our discussion of the Braid arrangement.

Proof: Using the definitions of s(l,k) and S(k,j) we have seen above we have:

$$\sum_{k=j}^{l} s(l,k)S(k,j) = \sum_{k=j}^{l}(\sum_{\substack{\pi\in\hat{\Pi}_l \\ |\pi|=k}} \mu(\hat{0},\pi)) \cdot (\#\{\lambda \in \Pi_l : |\lambda| = j, \lambda \geq \pi\})$$

$$= \sum_{\substack{\lambda\in\Pi_l \\ |\lambda|=j}} \sum_{\pi\leq\lambda} \mu(\hat{0},\pi)\zeta(\pi,\lambda)$$

$$= \sum_{\substack{\lambda\in\Pi_l \\ |\lambda|=j}} \mu\zeta(\hat{0},\lambda)$$

$$= \delta_{jl} = \begin{cases} 0 & \text{if } j \neq l \\ 1 & \text{if } j = l \end{cases}$$

□

2 Counting Connected components in $\mathbb{R}^l$

We now wish to take our study of hyperplanes to the real numbers. Suppose we have the hyperplane:

$$\mathcal{A} = \{H_1, ..., H_n\}, \text{defined over } \mathbb{Z}$$

and we let:

$$\gamma(\mathcal{A}) = \#\{\text{connected components of } \mathbb{R}^l - \cup_{i=1}^{n} H_i\}.$$

We will study how $\gamma(\mathcal{A})$ is related to the characteristic polynomial of the hyperplane $\mathcal{A}$. We will see $\gamma(\mathcal{A}) = |\chi(\mathcal{A}, -1)|$, which can be proved by showing the sides satisfy the same recurrence. Before we complete this proof we must introduce a few definitions and lemmas that will be useful.

Definition 3 *The deletion of $\mathcal{A}$, which we call $\mathcal{A}'$, is:*

$$\mathcal{A}' = \{H_1, ..., H_{n-1}\}.$$

That is, it is the set of $(n-1)$ hyperplanes in $\mathbb{R}^l$.

Definition 4 *The restriction of $\mathcal{A}$, which we call $\mathcal{A}''$, is:*

$$\mathcal{A}'' = \{H_1 \cap H_n, ..., H_{n-1} \cap H_n\}.$$

That is, it is a set of $(n-1)$ hyperplanes in $\mathbb{R}^{l-1}$.

Lemma 5 *Given a hyperplane $\mathcal{A}$ with deletion $\mathcal{A}'$ and restriction $\mathcal{A}''$ we have:*

$$\chi(\mathcal{A}, q) = \chi(\mathcal{A}', q) - \chi(\mathcal{A}'', q).$$

Proof: To prove this lemma we look at sets created by our hyperplanes. We have:

$$(\mathbb{F}_q^l - \cup_{i=1}^{n-1} H_i) = (\mathbb{F}_q^l - \cup_{i=1}^{n} H_i) \sqcup (H_n - \cup_{i=1}^{n-1}(H_i \cap H_n)).$$

Examining the cardinalities of each part here we find:

$$\chi(\mathcal{A}', q) = \chi(\mathcal{A}, q) + \chi(\mathcal{A}'', q).$$

This is equivalent to the statement in our lemma. □

We may be interested why there is no q in our recurrence. This is because q enters the equation only through the base case. For the base case of an empty arrangement ϕ_l in $\mathbb{F}_q^l$ we have:

$$\chi(\phi_l,) = \#\mathbb{F}_q^l = q^l.$$

Next we examine $\gamma(\mathcal{A})$.

Lemma 6 *Given a hyperplane $\mathcal{A}$ with deletion $\mathcal{A}'$ and restriction $\mathcal{A}''$ we have:*

$$\gamma(\mathcal{A}) = \gamma(\mathcal{A}') + \gamma(\mathcal{A}'').$$

Proof: We observe that each connected component of $H_n - \cup_{i=1}^{n-1}(H_i \cap H_n)$ partitions a connected component of $\mathbb{R}^l - \cup_{i=1}^{n-1} H_i$ into two parts. This means:

$$\gamma(\mathcal{A}) = \#\{\text{components of } \mathbb{R}^l - \cup_{i=1}^{n-1}(H_i)\}$$

$$= \#\{\text{components of } (\mathbb{R}^l - \cup_{i=1}^{n-1} H_i)\} + \#\{\text{components of } (H_n - \cup_{i=1}^{n-1}(H_i \cap H_n))\}$$

$$= \gamma(\mathcal{A}') + \gamma(\mathcal{A}'').$$

This completes our proof. □

We are now prepared to present our relation between $\chi(\mathcal{A}, q)$ and $\gamma(\mathcal{A})$.

Theorem 7 *If we have $\mathcal{A} = \{H_1, ...H_n\}$ defined over $\mathbb{Z}$ and $\gamma(\mathcal{A})$ as defined above then:*

$$\gamma(\mathcal{A}) = |\chi(\mathcal{A}, -1)|.$$

Proof: To begin our proof we define:

$$\tilde{\gamma}(\mathcal{A}) = (-1)^{dim\mathcal{A}}\gamma(\mathcal{A}).$$

We will seek to prove that $\tilde{\gamma}(\mathcal{A})$ equals $\chi(\mathcal{A}, q)$ by showing that they satisfy the same recurrence (with the same base case). We have already seen the recurrence satisfied by $\chi(\mathcal{A}, q)$ in Lemma 5 so all we need to study is $\tilde{\gamma}(\mathcal{A})$. From Lemma 6 we have:

$$\tilde{\gamma}(\mathcal{A}) = (-1)^{dim\mathcal{A}}\gamma(\mathcal{A}).$$

$$= (-1)^{dim\mathcal{A}}(\gamma(\mathcal{A}') + \gamma(\mathcal{A}'')).$$

Because $dim\mathcal{A}' = dim\mathcal{A}$ and $dim\mathcal{A}'' = dim\mathcal{A} - 1$ we can write this as:

$$\tilde{\gamma}(\mathcal{A}) = (-1)^{dim\mathcal{A}'}\gamma(\mathcal{A}') - (-1)^{dim\mathcal{A}''}\gamma(\mathcal{A}'').$$

$$= \tilde{\gamma}(\mathcal{A}') - \tilde{\gamma}(\mathcal{A}'').$$

Comparing this to the results of Lemma 5 it is thus clear that $\tilde{\gamma}(\mathcal{A})$ and $\chi(\mathcal{A}, q)$ satisfy the same recurrence equation so to prove they are equal we just need to show they have the same base case. For the base case we consider the empty arrangement, ϕ_l. For $\tilde{\gamma}$ we have:

$$\tilde{\gamma}(\phi_l) = (-1)^l \cdot 1.$$

We have seen $\chi(\phi_l, q) = q^l$ so considering $q = -1$ we have:

$$\chi(\phi_l, -1) = (-1)^l.$$

We then conclude that because $\tilde{\gamma}(\mathcal{A})$ and $\chi(\mathcal{A}, q)$ satisfy the same recurrence equation with the same base case that:

$$\tilde{\gamma}(\mathcal{A}) = \chi(\mathcal{A}, q).$$

Considering absolute values we then have:

$$\gamma(\mathcal{A}) = |\tilde{\gamma}(\mathcal{A})| = |\chi(\mathcal{A}, q)|.$$

This completes our proof. □

3 Graph Theory

We now move on to a study of graph theory. We begin with a few definitions.

Definition 8 *A graph, G, is defined as a set V of verices and a set E of edges, where* $E \subset V \times V$. *We often represent G by drawing the set V as a set of dots and drawing a line between two elements if there is a single edge that contains both vertices.*

Definition 9 *A proper coloring of G (with q colors) is a map:*

$$c : V \to [q],$$

satisfying $c(i) \neq c(j)$ for all edges $(i, j) \in E$.

Because we have been discussing hyperplanes we now take a look at the relation between hyperplanes and graphs.

Definition 10 *Given a graph, G, on vertex set $[n]$ and edge set E, the graphical arrangement $\mathcal{A}_G$ is the set of hyperplanes $\{x_i = x_j\}_{(i,j)\in E}$.*

There is also a relation between colorings of a graph, G, and the characteristic polynomial of $\mathcal{A}_G$. We have:

$$\#\{\text{proper colorings of } G \text{ with } q \text{ colors}\}$$
$$= \{(x_1, ..., x_n) \in \mathbb{F}_q^n | x_i \neq x_j foralledges(i, j) \in E\}$$
$$= \mathbb{F}_q^n - \cup_{e\in E} H_e$$
$$\chi(\mathcal{A}_G, q).$$

We call this the chromatic polynomial of G.

4 Hall's Marriage Theorem

To begin our study of Hall's Marriage theorem we first introduce a set of new definitions:

Definition 11 *A graph $G = (V, E)$ is called bipartite if it has a proper 2-coloring. This is equivalent to saying:*

$$V = X \sqcup Y,$$

such that there are no edges withing X and no edges within Y (i.e. $E \subset (X \times Y) \cup (Y \times X)$).

Definition 12 *A matching of a graph G is a set of edges $e_1, ..., e_k \in E$ such that $e_i \cap e_j = \varnothing, \forall i \neq j$.*

Definition 13 *A complete matching (or perfect matching) is a matching $e_1, ..., e_k$ such that $V = e_1 \cup ... \cup e_k$.*

Definition 14 *Given $A \subseteq V$ the neighborhood of A, $\Gamma(A)$ is defined as:*

$$\Gamma(A) = \{v \in V | \exists a \in A, \exists e \in E, \ s.t. \ e = (a, v)\}.$$

One interesting question that can be asked about bipartite graphs is when they will have a complete matching. If we have a graph $G = (X \sqcup Y, E)$ that has a complete matching $(x_1, y_1), ..., (x_n, y_n)$ then for any $A \subseteq X$ it is obvious we must have:

$$|\Gamma(A)| \geq |A|.$$

This is because at least $y_1, ..., y_k$ must be in $\Gamma(x_1, ..., x_k)$.

Theorem 15 *Hall's Marriage Theorem states that a bipartite graph $G = (X \sqcup Y, E)$ with $|X| = |Y|$ has a complete matching if and only if:*

$$|\Gamma(A)| \geq |A|, \ \ \forall A \subseteq X.$$

Proof of this theorem is provided in the next lecture.