18.312: Algebraic Combinatorics	Lionel Levine
Lecture 16	
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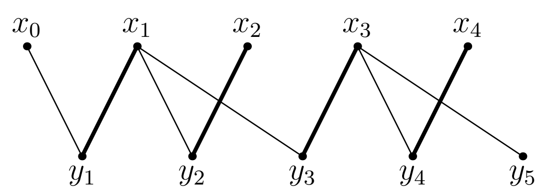
1 Matchings and Hall's Marriage Theorem

Theorem 1 (Hall) *Let $G = (V, E)$ be a finite bipartite graph where $V = X \cup Y$ with $X \cap Y = \emptyset$ and $|X| = |Y|$. Suppose that for all subsets $A \subset X$ we have $|\Gamma(A)| \geq |A|$ (recall that $\Gamma(A) = \{y \in Y \mid (x, y) \in E$ for some $x \in A\}$). Then G has a perfect matching (or complete matching).*

(Alternatively, we can remove the condition that $|X| = |Y|$ and change the conclusion to say that G has a matching which involves every vertex of X.)

Proof: Given a partial matching M with m edges, we will produce a partial matching M' with $m + 1$ edges. It is enough to find a path $x_0 y_1 x_1 \ldots y_k x_k y_{k+1}$ with $x_0 \notin H$, $y_{k+1} \notin M$, and $(y_i, x_i) \in M$ for $i = 1, 2, \ldots, k$. Given such a path, the set of edges $M' = \big(M \setminus \{(y_i, x_i)\}_{i=1}^{k}\big) \cup \{(x_i, y_{i+1})\}_{i=0}^{k}$ is a matching where $|M'| = |M| + 1$.



To construct the path, suppose that there exists some $x_0 \in X$ which is unmatched in M. The condition $|\Gamma(\{x_0\})| \geq 1$ implies that there exists $y_1 \in Y$ such that $(x_0, y_1) \in E$. If y_1 is unmatched in M, then we have a path $x_0 y_1$ with the desired properties. Otherwise, there exists $x_1 \in X \setminus \{x_0\}$ such that $(y_1, x_1) \in M$; the condition $|\Gamma(\{x_0, x_1\})| \geq 2$ implies that there exists $y_2 \neq y_1$ such that $(x_{r(2)}, y_2) \in E$ where $r(2)$ is either 0 or 1. In general, given $\{x_0, x_1, \ldots, x_{i-1}\}$ we can find some $y_i \notin \{y_1, \ldots, y_{i-1}\}$ such that $(y_i, x_{r(i)}) \in E$ for some $r(i) \in \{0, 1, \ldots, i-1\}$. This process of finding new y_i must terminate since Y is finite. We have constructed a set $\{x_0, y_1, x_1, \ldots, y_{l-1}, x_{l-1}, y_l\}$ such that $(y_i, x_i) \in M$ for all M, $x_0 \notin M$, and $y_l \notin M$ by construction. However, x_i, y_{i+1} may not be an edge for some i. To this end we take the subset $y_l, x_{r(l)}, y_{r(l)}, x_{r^2(l)}, y_{r^2(l)}, \ldots$ which must terminate with the last two terms y_1, x_0 since $r(1) = 0$ and $r^n(k) > r^{n+1}(k)$ for all n. In the above diagram, the desired path is $y_5 x_3 y_3 x_1 y_1 x_0$. $\square$

Theorem 2 (Kőnig) *Given a rectangular* $0-1$ *matrix* $M = (a_{ij})$ *where* $1 \leq i \leq m$ *and* $1 \leq j \leq n$*, define a "line" of* M *to be a row or column of* M*. Then the minimum number of lines containing all* $1s$ *of* M *is equal to the maximum number of* $1s$ *in* M *such that no two lie on the same line.*

Proof: Define a bipartite graph $G = (V, E)$ where $V = X \cup Y$, X is the set of rows of M, Y is the set of columns of M, and $(r_i, c_j) \in E$ if and only if $a_{ij} = 1$ (where r_i and c_j are arbitrary elements of X and Y, respectively). This allows us to restate Kőnig's Theorem as follows. A **vertex cover** of G is a set $C \subset V$ such that every edge $e \in E$ contains some element of C. Then

$$\min\{|C| \;:\; \text{vertex covers } C\} = \max\{|M| \;:\; \text{matchings } M\}.$$

Given any vertex cover C and any matching M, we have $|M| \leq |C|$ since C contains at least one vertex from each edge of M. Now it suffices to show that, given a minimal vertex cover C, we want to show that there exists a matching M such that $|M| = |C|$. Consider the graph $G' = (V, E')$ obtained by removing all the edges within C; $E' = E - (E \cap (C \times C))$. Then G' is bipartite with parts C and $V - C$ (no edges between C by construction, no edges between $V - C$ since C is a vertex cover).

We check Hall's condition for G'. Suppose there exists $A \subset C$ such that $|\Gamma(A)| < |A|$. The set $(C - A) \cup \Gamma(A)$ constitutes a vertex cover of G (thus contradicting the minimality of vertex cover C) unless there are edges in A that were removed by constructing G' from G. We will consider this case next lecture. □

Definition 3 *A* ***permutation matrix*** P *is a matrix whose entries are*

$$p_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{else} \end{cases}$$

for some $\sigma \in S_n$*.*

Theorem 4 (Birkhoff) *Let* $k.n \in \mathbb{N}$ *and let* $M = (a_{ij})_{i,j=1}^{n}$ *be an* $n \times n$ *matrix where its entries* a_{ij} *are nonnegative integers satisfying*

$$\sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = k.$$

Then there exist permutation matrices $p_1, \ldots, p_k$ *such that* $M = p_1 + \ldots + p_k$*.*

Proof: We proceed by induction on k. Consider the graph $G = (V, E)$ with $V = \{1, \ldots, n\} \cup \{1', \ldots, n'\}$ where i represents the $(i, j') \in E$ if and only if $a_{ij} \geq 1$. For all subsets $A \subset [n]$

we have

$$\sum_{j=1}^{n}\sum_{i\in A} a_{ij} = \sum_{i\in A}\sum_{j=1}^{n} a_{ij} = \sum_{i\in A} k = k|A|$$

and also for some fixed j we have

$$s_j := \sum_{i\in A} a_{ij} \leq \sum_{i=1}^{n} a_{ij} = k$$

so at least $|A|$ of the s_j are greater than 0. Since $j \in \Gamma(A)$ if and only if $\sum_{i\in A} a_{ij} > 0$, so $|\Gamma(A)| \geq |A|$. By Hall's Theorem, G has a perfect matching; therefore, there exists $\sigma \in S_n$ such that $(i, \sigma(i)') \in E$ for all $i = 1, 2, \ldots, n$. So the permutation matrix P corresponding to this permutation σ satisfies $p_{ij} \leq a_{ij}$ for all i, j. Now consider the matrix $M - P = (b_{ij})$;

$$\sum_{i=1}^{n} b_{ij} = \sum_{i=1}^{n} a_{ij} - \sum_{i=1}^{n} p_{ij} = k - 1.$$

By the induction hypothesis, we can write $M - P$ as the sum of permutation matrices; hence M is the sum of permutation matrices. □