18.312: Algebraic Combinatorics — Lionel Levine

Lecture 23

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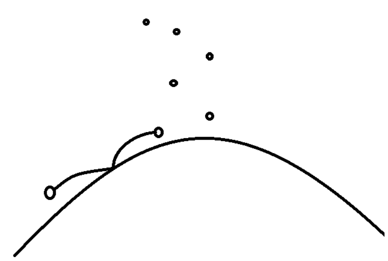
1 Abelian Sandpile Model

Definition 1 (sink, sandpile, chips) *Let $G = (V,E)$ be a finite connected undirected graph with a sink vertex z. Then a sandpile is a map $\sigma : V \to \mathbb{Z}_{\geq 0}$ such that $\sigma(v)$= number of chips at vertex v.*

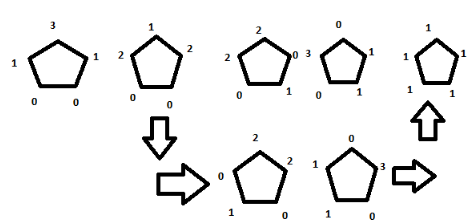
Definition 2 (stable/unstable) *A vertex v, which is not a sink, is unstable if $\sigma(v) \geq d(v)$ where $d(v)$ is the degree of v. Otherwise is is stable.*

Definition 3 (topple) *If v is unstable, it can topple, giving rise to a new sandpile $\sigma'(w) = \sigma(v) - d(v)$ for $w = v$, $\sigma(w) + 1$, when w and v are neighbors, and $\sigma(w)$ otherwise.*

The inspiration behind these names comes from the following example. Consider grains of sand being droped onto a sandpile. As the sand in the pile increases there is a trickling down effect which corresponds to toppling. See the image below.



Now consider the following exapmle where $G = C_5$.



Notice that toppling vertics in a different order leads to the same final result. We will prove a theorem about this after developing a few more definitions.

If $v \in V$, let $\Delta_v(w) = -d(v)$, $w = v$, 1 if w is neighbor of v, and 0 otherwise. Then toppling vertex v corresponds to setting $\sigma' = \sigma + A_v$.

Definition 4 (legal sequence) *A sequence $x_1, \ldots, x_k \in V$ is a* <u>*legal*</u> *toppling sequence for if $\sigma_i(x_i) \geq d(x_i)$ for all $i = 1, \ldots, k$ where $\sigma_1 = \sigma$ and $\sigma_{i+1} = \sigma_i + \Delta_{x_i}$ for $i = 1, \ldots, k-1$.*

Definition 5 (stabilizing) *A sequence $y_1, \ldots, y_l$ is* <u>*stabilizing*</u> *for σ if $(\sigma + \Delta_{y_1} + \cdots + \Delta_{y_l})w \leq d(w) - 1$ for all $w \in V - \{z\}$, where z is the sink.*

In the pentagon example above, we see thta (1,2,1,5),(1,5,1,2),(1,5,2,1), and (1,2,5,1) are all legal and stabilizing for $\sigma = 3-1-0-0-1$. Notice that the legal and stabilizing sequences are all permutations of each other. This inspires the following theorem.

Theorem 6 (Abelian Property) *If $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$ are both legal and stabilizing for σ, then $k = l$, and there exists $\pi \in S_k$ such that $x_i = y_{\pi_i}$ for all $i = 1, \ldots, k$*

We will get more millage out of the next lemma which implies the abelian property.

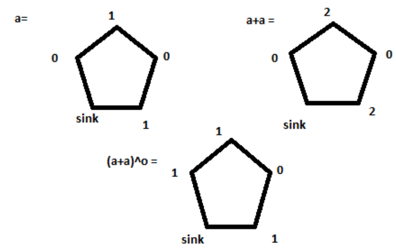
Lemma 7 *If $x_1, \ldots, x_k$ is legal for σ and $y_1, \ldots, y_l$ is stabilizing for σ, then $k \leq l$ and there exists $\pi \in S_l$ such that $x_i = y_{\pi_i}$ for all $i = 1, \ldots, k$.*

Proof: Induct on k. $\bar{x}$ legal implies $\sigma(x_1) \geq d(x_1)$. Notice that toppling any vertex $v \neq x$ does not decrease $\sigma(x_1)$. $\bar{y}$ stabilizing implies that there exists j such that $y_j = x_1$. Set $\pi(i) = j$. The sequence $y_j, y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_l$ is stabilizing for σ which implies $y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_l$ is stabilizing for $\sigma_2 = \sigma_1 + \Delta_{x_1}$. Also $x_2, \ldots, x_k$ is legal for σ_2. By inductive hypothesis, $x_2, \ldots, x_k$ is a permutation of a subsequence of $y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_l$. $\square$

Definition 8 (sandpile monoid) *The* <u>*sandpile monoid*</u> *of (G, z) is $M(G, z) = \{$stable sandpiles $\sigma : V_o \to Z_{\geq 0}\}$, where $V_o = V - \{z\}$, with operation $(\sigma_1, \sigma_2) \to (\sigma_1 + \sigma_2)^o$*

Definition 9 (stabilization) *The stabilization σ^o of σ is $\sigma^o = \sigma + \Delta_{x_1} + \cdots + \Delta_{x_k}$ where $x_1, \ldots, x_k$ is a legal stabilizing sequence for σ.*

See the example below.



Now M is a monoid because it is associative: $((\sigma_1 + \sigma_2)^o + \sigma_3)^o = (\sigma_1 + \sigma_2 + \sigma_3)^o = (\sigma_1 + (\sigma_2 + \sigma_3)^o)^o$ and has an identity: $(\sigma + 0)^o = \sigma$.

Theorem 10 *Let M be a finite commutative monoid. Then $J = \bigcap_{I \subset M,\ I\ ideal} I$ is an abelian group.*

Proof: Given $x \in J$, claim $x + J = J$, ie. $J \to J$ and $y \to x + y$ is a permutation of J. First note that J itself is an ideal: $J + M = \cap_{I \subset M}(I + M) \subseteq \cap I = J$. Now will show x+J is an ideal. $(x + J) + M = x + (J + M)x + J$ and since J is minimal ideal $J \subset x + J$. Hence $x + J = J$. Now will show the existence of an identity element. Let $\pi_x : J \to J$ be a permutation with $\pi_x(y) = x + y$. The exists $n \geq 1$ such that $\pi_x^n(y) = y$. Then nx+y = y for all $y \in J$ and we can let e= nx. Then e+y = y for all $y \in J$ and e is unique because $e = e + e' = e'$. e is our identity. Now define -x = (n-1)x so x+(-x) = nx = e. Then -x is inverse of x. Hence this is an abelian group.

□

Definition 11 (sandpile group) *The* <u>*sandpile group*</u> *of (G, z) is* $K(G, z) = \bigcap_{I \subset M(G,z), I\ ideal} I$.

For the last ten minutes of lecture Professor Levine showed some examples of sandpiles and talked about research in the field.