Lecture 24

Lecture date: May 12, 2011 *Notes by: Christopher Policastro*

Sandpile groups and Laplacian matrices

Let $G = (V, E)$ be a connected undirected graph. Enumerate the vertices as $V = \{v_1, \ldots, v_n\}$, and set $s = v_n$. Recall that a sandpile on G is a map $\sigma : V \to \mathbb{Z}_{\geq 0}$ or equivalently an element of $\mathbb{Z}^n_{\geq 0}$. The sandpile monoid on G is the commutative monoid $M(G,s) = \{\sigma : \sigma(v_i) < \deg(v_i), 1 \leq i \leq n-1\}$ with composition $\sigma +_{M(G,s)} \psi = (\sigma + \psi)^\circ$. The sandpile group on G is the abelian group

$$\kappa(G,s) = \bigcap_{I \text{ ideal of } M(G,s)} I.$$

This definition is slightly unworkable. We would like to find generators and relations for $\kappa(G,s)$ because abelian groups are usually understood in these terms.

Definition 1 *Let σ and ψ be sandpiles. If there exists a sandpile φ such that $\varphi(v_i) \geq 0$ for $1 \leq i \leq n-1$ and $\sigma = (\varphi + \psi)^\circ$, then σ is* ***reachable*** *from ψ. A sandpile $\sigma \in M(G,s)$ is* ***recurrent*** *if σ is reachable from any $\psi \in M(G,s)$.*

Let $R \subset M(G,s)$ be the set of recurrent sandpiles. Consider the sandpile $\delta^\circ \in M(G,s)$ such that $\delta(v_i) = \deg(v_i)$ for $1 \leq i \leq n-1$. Since $\delta(v_i) - \psi(v_i) > 0$ for any $\psi \in M(G,s)$, this implies that δ° is reachable. Hence $R \neq \emptyset$.

Lemma 2 *$\sigma \in \kappa(G,s)$ if and only if σ is recurrent.*

Proof: ($R \subset \kappa(G,s)$). Let $I \subset M(G,s)$ be a nonempty ideal. Fix $\psi \in I$. Given $\sigma \in R$, there exists a sandpile φ such that $\sigma = (\varphi + \psi)^\circ$ by defintion. Since I is an ideal, this implies that $\sigma \in I$. Hence $R \subset I$. We conclude that $R \subset \bigcap_{I \text{ ideal of } M(G,s)} I = \kappa(G,s)$.

($\kappa(G,s) \subset R$). Recall that $R \neq \emptyset$. Since $\kappa(G,s)$ is the minimal ideal of $M(G,s)$, it is enough to show that R is an ideal. Consider $\sigma \in R$ and $\tau \in M(G,s)$. We want to show that $(\sigma+\tau)^\circ$ is recurrent. Let $\psi \in M(G,s)$ and choose a sandpile φ such that $\sigma = (\psi+\varphi)^\circ$. Set $\varphi' = \varphi + \tau$. By the abelian property of sandpile stabilization, we have

$$(\psi + \varphi')^\circ = (\psi + \varphi + \tau)^\circ = ((\psi + \varphi)^\circ + \tau)^\circ = (\sigma + \tau)^\circ.$$

Hence $(\sigma + \tau)^\circ$ is recurrent. $\square$

Recall that the Laplacian matrix of G is the matrix $L = D - A$ where $D = \mathrm{diag}(\deg(v_1), \ldots, \deg(v_n))$ and $A = (a_{ij})_{i,j=1}^n$ for

$$a_{ij} = \begin{cases} 0 & \text{if } (v_i, v_j) \notin E \\ 1 & \text{if } (v_i, v_j) \in E \end{cases}.$$

Let L_s be the matrix L with the n^{th} row and n^{th} column removed, and let Δ_i for $1 \leq i \leq n-1$ be the i^{th} row of L_s.

Note that if σ is obtained from ψ through topplings, then $\sigma = \psi - \sum_{k=1}^m \Delta_{i_k}$ for some collection of vertices $\{v_{i_k} : 1 \leq k \leq m\} \subset V$. This is an immediate consequence of the definition of toppling. Hence $\sigma \sim \psi$ in $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$, and in particular $\sigma \sim \sigma^\circ$ in $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$. We will give an isomorphism between $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$ and $\kappa(G, s)$ by showing that each equivalence class in $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$ contains a unique recurrent sandpile and using our description of $\kappa(G, s)$ from Lemma 2.

Lemma 3 *Every equivalence class in $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$ contains at least one recurrent sandpile.*

Proof: Consider $\sigma \in \mathbb{Z}^{n-1}$. Let $m = \min\{0, \min\{\sigma(v_i) : 1 \leq i \leq n-1\}\}$ and $d = \max\{\deg(v_i) : 1 \leq i \leq n\}$. Recall the definition of δ. Set

$$\psi = \sigma + [d - m](\delta - \delta^\circ).$$

Since $\delta^\circ(v_i) < \deg(v_i)$ for $1 \leq i \leq n-1$, this implies that $\delta - \delta^\circ$ is a positive vector. Hence $d - m \geq -m$ implies that ψ is a nonnegative vector. Since $\delta - \delta^\circ \sim 0$ in $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$, this means that $\psi \sim \sigma$ in $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$. We claim that $\psi^\circ \in M(G, s)$ is recurrent. Note that

$$\psi(v_i) \geq [d - m](\delta(v_i) - \delta^\circ(v_i)) \geq d(\delta(v_i) - \delta^\circ(v_i)) \geq d$$

for $1 \leq i \leq n-1$. Given $\tau \in M(G, s)$, $\psi(v_i) \geq \deg(v_i)$ for $1 \leq i \leq n-1$ implies that $\psi - \tau$ is a nonnegative vector. Hence $\psi^\circ = (\tau + (\psi - \tau))^\circ$. We conclude that ψ° is a recurrent sandpile equivalent to σ in $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$. $\square$

Fact 4 *Set $\epsilon = (2\delta) - (2\delta)^\circ$. If σ is recurrent, then $(\sigma + \epsilon)^\circ = \sigma$.*

Proof: By definition there exists a sandpile τ such that $\sigma = (\tau + \delta)^\circ$. Consider the sandpile

$$\psi = (\tau + \delta) + \epsilon = (2\delta) + \tau + \delta - (2\delta)^\circ.$$

Since ϵ is a positive vector, the abelian property of sandpile stabilization implies that

$$\psi^\circ = ((\tau + \delta)^\circ + \epsilon)^\circ = (\sigma + \epsilon)^\circ.$$

Again since $\delta - (2\delta)^\circ$ is a nonnegative vector, the abelian property of sandpile stabilization implies that

$$\psi^\circ = ((2\delta)^\circ + \tau + \delta - (2\delta)^\circ)^\circ = (\tau + \delta)^\circ = \sigma.$$

This gives the result. □

Lemma 5 *Every equivalence class in* $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$ *contains at most one recurrent sandpile.*

Proof: Suppose that σ and σ' are recurrent and equivalent in $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$. This implies that

$$\sigma' = \sigma + \sum_{k \in I_+} c_k \Delta_{i_k} + \sum_{k \in I_-} c_k \Delta_{i_k}$$

where $c_k > 0$ for $k \in I_+$ and $c_k < 0$ for $k \in I_-$. Let

$$\psi = \sigma' + \sum_{k \in I_-} -c_k \Delta_{i_k} = \sigma + \sum_{k \in I_+} c_k \Delta_{i_k}.$$

Recall the definition of ϵ. Since ϵ is a positive vector, there exist $N \gg 0$ such that $\tau(v_k) \geq |c_k| \deg(v_i)$ for $\tau = \psi + N\epsilon$. Topple each vertex v_k for $k \in I_-$ in τ a total of $-c_k$ times to obtain $\sigma' + k\epsilon$. By Fact 4, $\sigma' + k\epsilon$ stabilizes to σ'. Topple each vertex v_k for $k \in I_+$ in τ a total of c_k times to obtain $\sigma + k\epsilon$. By Fact 4, $\sigma + k\epsilon$ stabilizes to σ. Therefore by the abelian property of sandpile stabilization, we conclude that $\sigma = \sigma'$. □

Assume that L_s has Smith normal form $UL_sV = \mathrm{diag}(b_1, \ldots, b_{n-1})$ where $U, V \in \mathrm{GL}_{n-1}(\mathbb{Z})$.

Theorem 6 $\kappa(G, s) \cong \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s \cong \mathbb{Z}/b_1\mathbb{Z} \times \cdots \times \mathbb{Z}/b_{n-1}\mathbb{Z}$.

Proof: By Lemma 2, Lemma 3 and Lemma 5 imply that the map taking a recurrent sandpile to its equivalence class in $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$ is bijective. Since $(\sigma + \tau) \sim (\sigma + \tau)^\circ$ for $\sigma, \tau \in \mathbb{Z}^n_{\geq 0}$ this map is a group homomorphism. This gives the first isomorphism.

Writing vectors as columns rather than rows, $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$ becomes $\mathbb{Z}^{n-1}/L_s^t\mathbb{Z}^{n-1}$. The second isomorphism follows from Example 6 of Lecture 22 noting that
$\mathrm{diag}(b_1, \ldots, b_{n-1}) = \mathrm{diag}(b_1, \ldots, b_{n-1})^t = V^t L_s^t U^t$. □

Theorem 6 gives the sought after description of $\kappa(G, s)$ in terms of generators and relations. Recall that $\kappa(G)$ is the number of spanning trees in G.

Corollary 7 $|\kappa(G, s)| = b_1 \cdots b_{n-1} = \kappa(G)$

Proof: By Theorem 6, we have that

$$|\kappa(G,s)| = |\mathbb{Z}/b_1\mathbb{Z} \times \cdots \times \mathbb{Z}/b_{n-1}\mathbb{Z}| = b_1 \cdots b_{n-1} = \det(L_s).$$

Note that we have used the fact that $\det(U) = \det(V) = \pm 1$. From the Matrix Tree Theorem, we know that $\det(L_s)$ is the number of spanning trees rooted at s in the bidirected graph corresponding to G, or equivalently the number of spanning trees in G. This gives the result. □

The definition of $\kappa(G,s)$ makes the dependence on s unclear. However using Theorem 6 we see that the choice of the sink is irrelevant.

Corollary 8 *For any $s' \in V$, $\kappa(G,s) \cong \kappa(G,s')$.*

Proof: By Theorem 6, we know that $\kappa(G,s') \cong \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_{s'}$. We claim that $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_{s'}$ is isomorphic to $\mathbb{Z}^n/\mathbb{Z}^n L$ for each $s' \in V$. Reordering if necessary, assume without loss of generality that $s' = s$. Note that $\mathbb{Z}^{n-1}$ is isomorphic to the subgroup of vectors in $\mathbb{Z}^n$ whose coordinates sum to zero. Since the rows of L sum to zero, this isomorphism is compatible with quotienting $\mathbb{Z}^n$ by $\mathbb{Z}^n L$. In other words, modding out a vector in $\mathbb{Z}^{n-1}$ by the $\mathbb{Z}$-span of the rows of L_s corresponds to modding out the related vector in $\mathbb{Z}^n$ by the $\mathbb{Z}$-span of the rows of L. Since the rows of L sum to zero, the $\mathbb{Z}$-span of rows 1 to n of L is the same as the $\mathbb{Z}$-span of rows 1 to $n-1$ of L. We conclude that $\mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}L_s$ is isomorphic to $\mathbb{Z}^n/\mathbb{Z}^n L$. □

Action of sandpile groups on spanning trees

We want to extend Corollary 7 by producing a bijection between $\kappa(G,s)$ and the spanning trees of G. Let G' be the bidirected graph corresponding to G with the edges coming out from s removed. Let T denote the set of oriented spanning trees rooted at s in G'. Recall that T is in bijection with the set of spanning trees in G. The set T does not have an obvious group structure. Even the composition of spanning trees is unclear. So assigning T a group structure and producing an isomorphism is not a reasonable plan.

A better idea is to find a ***free*** and ***transitive*** action of $\kappa(G,s)$ on T. Recall that the action of a group on a set is free if only the identity element has a fixed point, and transitive if there exists a single orbit. Hence for any $t, t' \in T$ there would exist a unique element $\sigma \in \kappa(G,s)$ such that $\sigma t = t'$. Such an action can be given in terms of rotor-routing, which was described in Lecture 20.

Fix an ordering E_i on the edges incident to v_i for $1 \le i \le n-1$. A rotor configuration on (G,s) is a map $\rho : V - \{s\} \to E$ such that $\rho(v_i) \in E_i$ for $1 \le i \le n-1$. Consider a sandpile σ and a rotor configuration ρ. A non-sink vertex of G is ***active*** if it has at least one chip. It v_i is active then ***firing*** v_i results in a new sandpile and rotor configuration given by replacing the rotor $\rho(v_i)$ with $\rho(v_i)^+$ and moving one chip from v_i to the head of $\rho(v_i)^+$ (and removing the chip if $\rho(v_i)^+$ is a sink).

Let $\sigma \in \kappa(G, s)$ and $t \in T$. The action of σ on t can be described by the following process. The edges of t determine a rotor configuration. Place $\sigma(v_i)$ chips at v_i for $1 \leq i \leq n-1$. Fire the vertices of G until no vertex is active. The resulting rotor configuration determines an element $t' \in T$. The image of t under σ is t'. Showing that this process determines a well defined action, and that this action is free and transitive takes some developing. We refer the interested reader to Section 3 of Holroyd et al..

Example: Let G and G' be the graphs indicated in Figure 1. Throughout the example we will consider sandpiles on G' which are conceptually identical to sandpiles on G. There

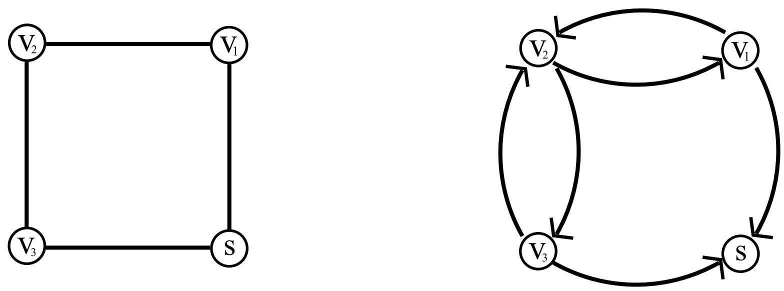

Figure 1: Graphs G and G'

exist four spanning trees in G' rooted at s. These are shown in Figure 2.

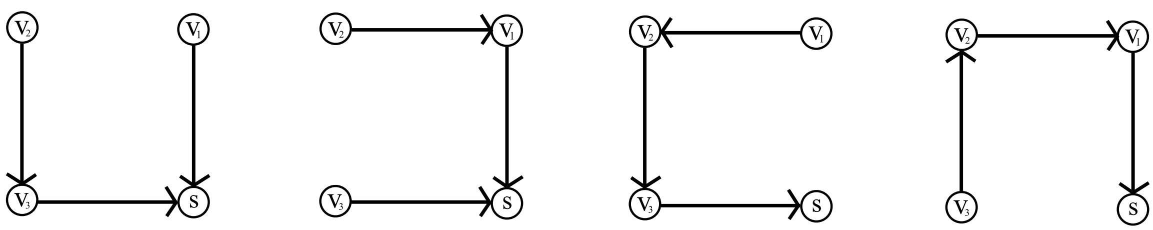

Figure 2: Spanning trees in G' rooted at s

Since vertices v_1, v_2 and v_3 have outdegree 2, $M(G', s)$ consists of the 8 vectors (x, y, z) for $x, y, z \in \{0, 1\}$. To determine which of these sandpiles are recurrent we can use the following lemma whose proof can be found in Section 4 of Holroyd et al.

Lemma 9 (Burning Algorithm) *Let β be the sandpile on G' such that*

$$\beta(v_i) = \text{outdeg}(v_i) - \text{indeg}(v_i) \geq 0.$$

A sandpile σ is recurrent if and only if $(\sigma + \beta)^\circ = \sigma$.

Using Lemma 9, we find that there exist 4 recurrent sandpiles. These are indicated in Figure 3. Moreover $\kappa(G', s) \cong \mathbb{Z}/4\mathbb{Z}$ where the isomorphism is given in Figure 3. Let σ be the

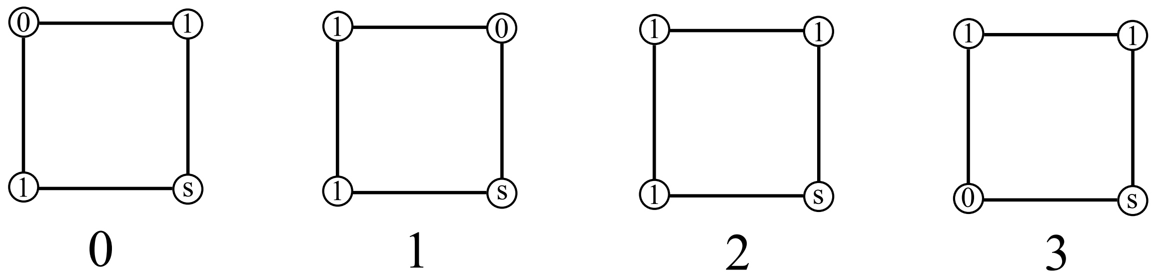

Figure 3: Recurrent sandpiles on G'

sandpile $(1, 1, 0)$, and let t be the spanning tree indicated in Figure 4. Using the procedure outlined above, we find that the action of σ on t is given by the sequence of spanning trees in Figure 4. Note that $\sigma t = t'$ is indeed a spanning tree in G'.

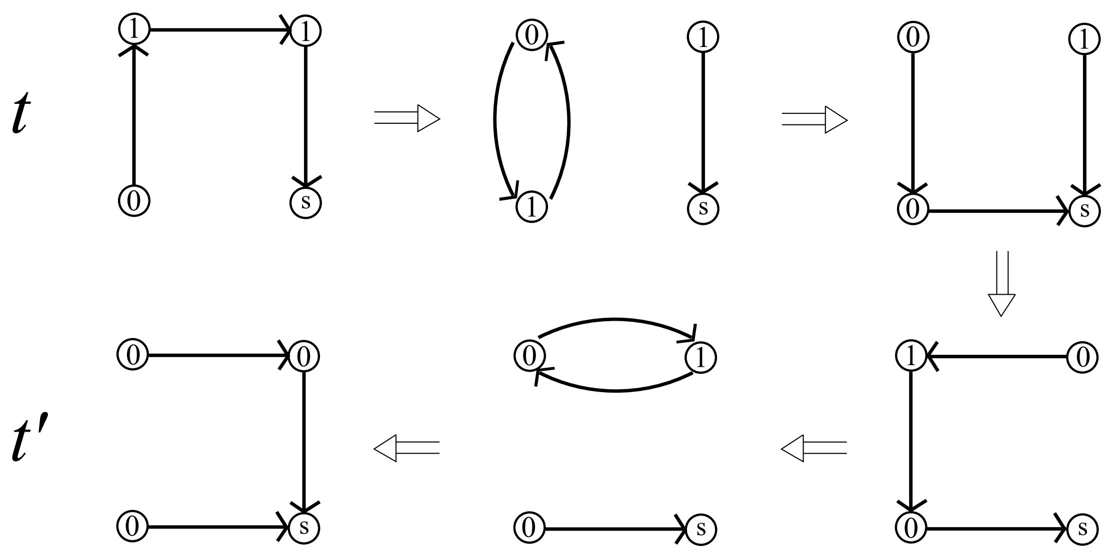

Figure 4: Action of σ on t

Complete graphs and parking sequences

The sandpile group of K_n, the complete graph on n vertices, is already of interest. Recall that the Laplacian matrix of K_n is $nI_n - J$ where I_n is the $n \times n$ identity and J is the $n \times n$ matrix whose entries are all 1. Fix a sink s and let L_s be the corresponding reduced

Laplacian matrix. Recall that the (k,k) entry of the Smith normal form of L_s is the gcd of the $k \times k$ minors of L_s. For instance $b_1 = 1$, and b_2 is the gcd of

$$\begin{vmatrix} n-1 & -1 \\ -1 & n-1 \end{vmatrix} \quad \begin{vmatrix} n-1 & -1 \\ -1 & -1 \end{vmatrix} \quad \begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix}$$

which is n. It can be shown that the gcd of the $k \times k$ minors for $k \geq 2$ is n. By Theorem 6 this implies the following result.

Theorem 10 $\kappa(K_n, s) \cong \left(\mathbb{Z}/n\mathbb{Z}\right)^{n-2}$.

Note that by Corollary 7, we have Cayley's formula $\kappa(K_n) = n^{n-2}$. From this result we can obtain a description of the recurrent sandpiles on K_n. A sandpile σ is recurrent if and only if at least k vertices contain at least $n-k$ chips for $1 \leq k \leq n$. This puts recurrent sandpiles on K_n in bijective correspondence with solutions to the following problem.

Consider n parking spaces labeled 1 to n along a one way street. See Figure 5. There exist n cars that want to park in these spaces. Each driver has a prefered spot. The drivers will take turns selecting a spot, and will take the next available spot if they find that their prefered spot has already been taken. The preferences of the drivers can be expressed as a sequence $s_i \in [n]$ for $1 \leq i \leq n$. A ***parking sequence*** is a sequence $\{s_i\}$ of preferences such that every driver finds a spot. Note that at most one driver can want spot n, at most two drivers can want spot $n-1$ and so on, establishing the bijection.

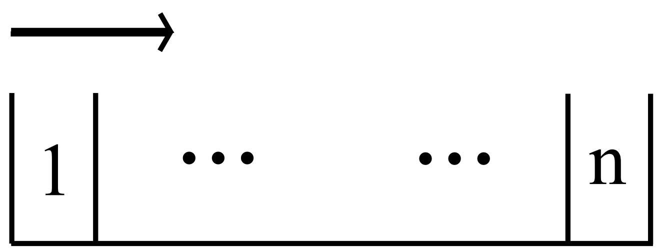

Figure 5: Setup in parking problem

References

A. Holroyd, L. Levine, K. Mészáros, Y. Peres, J. Propp, D. Wilson, "Chip-Firing and Rotor-Routing on Directed Graphs," arXiv:0801.3306v3.