18.312: Algebraic Combinatorics	Lionel Levine
Lecture 5	
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1 Stirling inverse matrices

From last class, we have the following proposition:

Proposition 1

$$\sum_{k=0}^{n} S(n,k)s(k,j) = \delta_{nj}$$

where $S(n,k)$ are Stirling numbers of the second kind, $s(k,j)$ are signed Stirling numbers of the first kind such that

$$s(k,j) = (-1)^{k-j}c(k,j),$$

and

$$\delta_{nj} = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{otherwise} \end{cases}$$

Example 2 *Consider the $n = 4$ case, in which S and s are 4×4 matrices:*

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{pmatrix} = s$$

Proof: Recall these two facts from last class:

Fact 3

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1)$$

Fact 4

$$\sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1) = x^n$$

First, we will find an expression analagous to fact 3 in terms of signed Stirling numbers of the first kind.

$$\begin{aligned}
\sum_{k=0}^{n} s(n,k)x^k &= \sum_{k=0}^{n} (-1)^{n-k} c(n,k)x^k \\
&= (-1)^n \sum_{k=0}^{n} c(n,k)(-x)^k \\
&= (-1)^n(-x)(-x+1)\cdots(-x+n-1) \quad \text{ by fact 3} \\
&= x(x-1)\cdots(x-n+1) \quad \text{ since we have one -1 per factor}
\end{aligned}$$

Now, let vector space $V_n = \{\text{polynomials in } x \text{ of degree} \leq n \text{ with constant term } 0\}$. Consider two bases for V_n:

$$e_i = x^i$$

and

$$f_i = x(x-1)\cdots(x-i+1)$$

for i from 1 to n.

Define $L : V_n \to V_n$ to be the linear operator such that $L(e_i) = f_i$.

From above, we know that

$$f_i = \sum_{k=0}^{i} s(i,k)e_k$$

so the matrix of L in the basis $e_1, \ldots, e_n$ is $(s(i,k))_{i,k=1}^{n}$.

Then by fact 4, the matrix of L^{-1} in the basis $f_1, \ldots, f_n$ is $(S(n,k))_{i,k=1}^{n}$.

That is,

$$\sum_{k=0}^{n} S(n,k) f_k = e_k.$$

Substituting in for f_k, we get

$$\sum_{k=0}^{n} S(n,k) \sum_{j=0}^{n} s(k,j)e_j = e_n.$$

Rearranging, we get

$$\sum_{j=0}^{n} \left(\sum_{k=0}^{n} S(n,k)s(k,j) \right) e_j = e_n.$$

Therefore,

$$\sum_{k=0}^{n} S(n,k)s(k,j) = \delta_{nj}.$$

□

2 Linear recurrences

Recall the following example from last class:

Example 5 *The Fibonacci sequence is defined by the recurrence*

$$F_{n+2} = F_{n+1} + F_n$$

for $n \geq 1$ with $F_1 = 1$ and $F_2 = 1$. We can write this recurrence in terms of the shift operator E as

$$(E^2 - E - 1)F = 0.$$

Factoring, we see that

$$(E - \phi)(E - \bar{\phi}) = 0$$

where

$$\phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots \quad \text{and} \quad \bar{\phi} = \frac{1-\sqrt{5}}{2} = -0.618\ldots.$$

We can then write

$$F_n = a\phi^n + b\bar{\phi}^n.$$

Using the initial conditions $F_1 = F_2 = 1$, we find that $a = \frac{1}{\sqrt{5}}$ and $b = -\frac{1}{\sqrt{5}}$, so

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \bar{\phi}^n).$$

Since $\bar{\phi}^n$ rapidly becomes very small, we can say that

$$F_n \approx \frac{\phi^n}{\sqrt{5}}.$$

For instance, $F_{10} = 55$ and $\frac{\phi^{10}}{\sqrt{5}} = 55.0036\ldots$.

Now, we wish to generalize these results. We wish to work over an algebraically closed field so we can factor. We will use $\mathbb{C}$.

Definition 6 *A sequence* $s = (s_0, s_1, s_2, \ldots) \in \mathbb{C}^\infty$ *obeys a linear recurrence of order* k *if there exist* $a_0, a_1, \ldots, a_{k-1} \in \mathbb{C}$ *such that*

$$s_{n+k} = \sum_{i=0}^{k-1} a_i s_{n+i}$$

for all $n \geq 0$.

We can therefore write linear recurrences in the form

$$p(E)s = 0$$

where p is a polynomial in $\mathbb{C}[x]$ of degree k.

Definition 7 *Suppose* p *factors as*

$$p(E) = (E - \phi_1) \ldots (E - \phi_k)$$

where $\phi_1, \ldots, \phi_k$ *are distinct complex numbers. Then* s *satisfies a simple linear recurrence.*

Theorem 8 *The sequence* s *satisfies the simple linear recurrence* $p(E)s = 0$ *if and only if there exist* $c_1, \ldots, c_k \in \mathbb{C}$ *such that*

$$s_n = c_1\phi_1^n + \ldots + c_k\phi_k^n.$$

That is, s_n *can be expressed as a linear combination of exponential sequences.*

Proof: $p(E) : \mathbb{C}^\infty \to \mathbb{C}^\infty$ is a linear operator. $\ker(p(E)) = \{s \mid p(E)s = 0\}$ is a subspace of $\mathbb{C}^\infty$. Let $e_n^{(i)} = \phi_i^n$. We want to show that $e^{(1)}, \ldots, e^{(k)}$ form a basis for $\ker(p(E))$.

First, we need to show that $e^{(i)} \in \ker(p(E))$. The $e^{(i)}$'s are eigenvectors of the shift operator.

$$(Ee^{(i)})_n = e_{n+1}^{(i)} = \phi_i^{n+1} = \phi_i\phi_i^n = \phi_i e_n^{(i)},$$

Thus, $Ee^{(i)} = \phi_i e^{(i)}$, so $e^{(i)}$ is an eigenvector of E with eigenvalue ϕ_i. This means that $e^{(i)} \in \ker(E - \phi_i)$, or $(E - \phi_i)e^{(i)} = 0$. Since multiplication commutes, we can write $p(E) = q(E)(E - \phi_i)$ for some polynomial q. Then we have that

$$\begin{aligned} p(E)e^{(i)} &= q(E)(E - \phi_i)e^{(i)} \\ &= q(E)0 \\ &= 0, \end{aligned}$$

so $e^{(i)} \in \ker(p(E))$.

Next, we show that $e^{(1)}, \ldots, e^{(k)}$ are linearly independent. Consider

$$\det\,(e_j^{(i)})_{i=1,j=0}^{k,k-1} = \det \begin{pmatrix} 1 & \cdots & 1 \\ \phi_1 & \cdots & \phi_k \\ \phi_1^2 & \cdots & \phi_k^2 \\ \vdots & \ddots & \vdots \\ \phi_1^{k-1} & \cdots & \phi_k^{k-1} \end{pmatrix}$$

This is the Vandermonde determinant, so we see that

$$\det\,(e_j^{(i)})_{i=1,j=0}^{k,k-1} = \prod_{i<j} (\phi_i - \phi_j).$$

Since we are dealing with a simple linear recurrence, all roots of p must be distinct, so this determinant is nonzero. This means that there is no linear dependence among the first k terms of the sequences, so there is no linear dependence among the sequences. Therefore, $e^{(1)}, \cdots, e^{(k)}$ are linearly independent.

Finally, to show that $e^{(1)}, \ldots, e^{(k)}$ form a basis, we need to show that $\dim(\ker(p(E))) = k$. A sequence $s \in \ker(p(E))$ is determined by its first k terms $s_0, \ldots, s_{k-1}$ since all subsequent terms $s_k, s_{k+1}, \ldots$ are determinted by the recurrence. Let $f_j^{(i)} = \delta_{ij}$ for $1 \leq i, j \leq k$. Then $f^{(1)}, \ldots, f^{(k)}$ form a basis of size k. All bases of $\ker(p(E))$ must have the same cardinality. Therefore, $e^{(1)}, \ldots, e^{(k)}$ form a basis for $\ker(p(E))$. $\square$

Example 9 *The sequence $s_n = 3^n - 2^n$ obeys a linear recurrence. Let $\phi_1 = 3$ and $\phi_2 = 2$. Then*

$$p(E) = (E-3)(E-2) = E^2 - 5E + 6,$$

so our recurrence is

$$s_{n+2} - 5s_{n+1} + 6s_n = 0.$$

What if $p(E)$ has repeated roots?

Example 10 *Consider the sequence s such that*

$$s_{n+3} = 3s_{n+2} - 3s_{n+1} + s_n.$$

Then

$$p(E) = E^3 - 3E^2 + 3E - 1 = (E-1)^3 = D^3,$$

where $D = E - 1$ is the difference operator. One solution is $s_n = 1^n = 1$. However, we expect to have two other linearly independent solutions since this a linear recurrence of order 3. These two additional solutions are $s_n = n$ and $s_n = n^2$.

D is analagous to the operator $\frac{d}{dt}$ for functions, so the corresponding differential equation to this recurrence is

$$\left[\frac{d}{dt}\right]^3 f(t) = 0.$$

Taking powers of the operator is the same as function composition, so this is equivalent to

$$\frac{d^3}{dt^3} f(t) = 0,$$

which has similar solutions $f(t) = 1$, $f(t) = t$, and $f(t) = t^2$.

This example brings us to the following lemma.

Lemma 11 *$D^m s = 0$ if and only if $s_n = q(n)$ for some polynomial q of degree less than or equal to $m - 1$.*

Proof: We want to show that $1, n, n^2, \ldots, n^{m-1}$ form a basis for $\ker(D_m)$. We know that $\dim(\ker(D^m)) = m$ since a sequence that satisfies linear recurrence of order m is determined by its first m terms, which can be chosen arbitrarily as described above.

We first show that $1, n, n^2, \ldots, n^{m-1}$ are linearly independent. If $1, n, n^2, \ldots, n^{m-1}$ were linearly dependent, then there would be c_i's not all equal to zero such that

$$\sum_{i=0}^{m-1} c_i n^i = 0 \text{ for all } n.$$

However, this would be a polynomial of finite degree with infinitely many roots. Therefore, all of the c_i's must be 0 and $1, n, n^2, \ldots, n^{m-1}$ must be linearly independent.

It remains to show that $1, n, n^2, \ldots, n^{m-1} \in \ker(D^m)$. We will prove this by induction on m. By our induction hypothesis, $1, n, n^2, \ldots, n^{m-2} \in \ker(D^{m-1})$, so

$$D^m[n^i] = D[D^{m-1}[n^i]] = D[0] = 0 \text{ for } i \leq m - 2.$$

We now need to show that $D^m[n^{m-1}] = 0$. We first find an expression for $D[n^{m-1}]$.

$$\begin{aligned} D[n^{m-1}] &= (E-1)n^{m-1} \\ &= (n+1)^{m-1} - n^{m-1} \\ &= \sum_{k=0}^{m-1} \binom{m-1}{k} n^k - n^{m-1} \quad \text{by the Binomial Theorem} \\ &= \sum_{k=0}^{m-2} \binom{m-1}{k} n^k \quad \text{Note that this is a polynomial of degree } m-2. \end{aligned}$$

Then we substitute this expression into $D^m[n^{m-1}]$:

$$\begin{aligned} D^m[n^{m-1}] &= D^{m-1}[D[n^{m-1}]] \\ &= D^{m-1}\left[\sum_{k=0}^{m-2} \binom{m-1}{k} n^k\right] \\ &= 0 \text{ by the inductive hypothesis since we are applying } D^{m-1} \text{ to a polynomial of degree } m-2 \end{aligned}$$

Therefore, $1, n, n^2, \ldots, n^{m-1} \in \ker(D^m)$ and $1, n, n^2, \ldots, n^{m-1}$ form a basis for $\ker(D_m)$. □

So far, we have looked at two special cases of linear recurrences: distinct roots and powers of the difference operator. We now consider the solution to a general linear recurrence.

Theorem 12 *The sequence $s = (s_0, s_1, s_2, \ldots)$ satisfies the linear recurrence*

$$\prod_{i=1}^{k} (E - \phi_i)^{m_i} s = 0$$

if and only if

$$s_n = q_1(n)\phi_1^n + \ldots + q_k(n)\phi_k^n,$$

where each q_i is a polynomial of degree at most $m_i - 1$.

The proof is similar to the proofs of the previous lemma and theorem. We will now continue to explore the connection between linear recurrences and differential equations.

3 Exponential generating functions

Definition 13 *Given a sequence $s = (s_0, s_1, s_2, \ldots)$, the exponential generating function of s is the power series*

$$\mathcal{F}_s(x) = s_0 + s_1 x + \frac{s_2 x^2}{2} + \ldots + \frac{s_n x^n}{n!} + \ldots.$$

Example 14 *Consider the sequence $s_n = 1$ for all n. Then*

$$\mathcal{F}_s(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Why is there a factorial in the denominator of each term?

$$\frac{d}{dx}\left[\frac{x^n}{n!}\right] = \frac{nx^{n-1}}{n!} = \frac{x^{n-1}}{(n-1)!},$$

so

$$\begin{aligned}\frac{d}{dx}[\mathcal{F}_s(x)] &= \frac{d}{dx}\left[s_0 + s_1 x + \frac{s_2 x^2}{2} + \ldots + \frac{s^n x^n}{n!} + \ldots\right] \\ &= s_1 + s_2 x + \ldots + \frac{s_n x^{n-1}}{(n-1)!} + \frac{s_{n+1} x^n}{n!} + \ldots \\ &= \mathcal{F}_{Es}(x).\end{aligned}$$

Differentiating an exponential generating function corresponds to shifting its sequence. In particular, if s obeys a linear recurrence $p(E)s = 0$, then its exponential generating function $\mathcal{F}_s(x)$ obeys the linear ordinary differential equation

$$p\left(\frac{d}{dx}\right)\mathcal{F}_s(x) = 0.$$

Why is this true?

$$p\left(\frac{d}{dx}\right)\mathcal{F}_s(x) = \mathcal{F}_{p(E)s}(x) = \mathcal{F}_0(x) = 0.$$

Example 15 *Consider this ordinary differential equation:*

$$f''(x) = f'(x) + f(x).$$

We can write $f(x)$ as

$$f(x) = \sum_{n=0}^{\infty} \frac{s_n x^n}{n!}$$

so finding s satisfying the linear recurrence

$$s_{n+2} = s_{n+1} + s_n$$

results in $f(x)$ satisfying the ODE. In this case, $s_n = F_n$, the Fibonacci Sequence, gives a solution.

Example 16 *Consider the power series for* $\sin x$. *We know that*

$$\frac{d^2}{dx^2}\sin x = -\sin x.$$

In operator notation, we can write this as

$$\left[\left[\frac{d}{dx}\right]^2 + 1\right]\sin x = 0.$$

We can write $\sin x$ *as*

$$\sin x = \sum_{n=0}^{\infty} \frac{s_n x^n}{n!}$$

where s *satisfies the recurrence*

$$s_{n+2} + s_n = 0$$

with $s_0 = 0$ *and* $s_1 = 1$ *since* $\sin 0 = 0$ *and* $\sin' 0 = \cos 0 = 1$. *The recurrence and initial values determine all of* s, *so we have that*

$$\sin x = 0 + 1x + 0\frac{x^2}{2} + -1\frac{x^3}{3!} + \dots.$$

Now, we will make a more explicit connection between the shift operator and the derivative.

4 Relating E and $\frac{d}{dx}$

We can think of applying the shift operator to functions in the following manner:

$$\begin{aligned}(Ef)(x) &= f(x+1)\\ (E^2 f)(x) &= f(x+2)\\ (E^h f)(x) &= f(x+h) \quad \text{for } h \in \mathbb{R}.\end{aligned}$$

Then we can write

$$\begin{aligned}\frac{df}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\\ &= \lim_{h\to 0} \frac{(E^h f)(x) - f(x)}{h}\\ &= \lim_{h\to 0} \left(\frac{E^h - 1}{h}\right) f \qquad \text{Things become less rigorous here.}\end{aligned}$$

By L'Hopital's Rule,

$$\begin{aligned}\lim_{h\to 0} \frac{a^h - 1}{h} &= \lim_{h\to 0} \frac{a^h \ln a}{1}\\ &= \ln a\end{aligned}$$

so, by analogy, we might say that

$$\lim_{h\to 0} \left(\frac{E^h - 1}{h}\right) f = (\ln E) f$$

so that

$$\frac{d}{dx} = \ln E.$$

There is a sense in which this is true, as we might then say based on knowledge of the Taylor series for e^x that

$$E = e^{\frac{d}{dx}} = 1 + \frac{d}{dx} + \frac{1}{2}\left(\frac{d}{dx}\right)^2 + \ldots + \frac{1}{n!}\left(\frac{d}{dx}\right)^n + \ldots.$$

This leads to

$$Ef = f(x+1) = f(x) + f'(x) + \frac{1}{2}f''(x) + \ldots + \frac{1}{n!}f^{(n)}(x) + \ldots,$$

which is indeed the Taylor series for $f(x+1)$ centered at $f(x)$.