18.312: Algebraic Combinatorics | Lionel Levine

Lecture 6

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1 Reprise of $\frac{d}{dx} = \ln(E)$

Let E denote the shift operator, such that for a sequence of numbers $s_0, s_1, s_2, \ldots$,

$$E(s_0, s_1, s_2, \ldots) = (s_1, s_2, s_3, \ldots).$$

In the previous lecture, we mentioned the equation

$$E = e^{\frac{d}{dx}}.$$

Also, as mentioned in the last lecture, we can also have E operate on functions. If f is a function, then let

$$(Ef)(x) = f(x+1).$$

We can also define E^h to be

$$(E^h f)(x) = f(x+h),$$

where h is any real number.

To better understand the equation $E = d^{\frac{d}{dx}}$, we recall the Taylor expansion of e^x.

$$e^t = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!} + \cdots$$

In a similar way, we can think of $e^{t\frac{d}{dx}}$ as

$$e^{t\frac{d}{dx}} = 1 + t\frac{d}{dx} + \frac{t^2\left(\frac{d}{dx}\right)^2}{2} + \cdots + \frac{t^n\left(\frac{d}{dx}\right)^n}{n!} + \cdots \tag{1}$$

In (1) above, multiplication of operators is the same as the composition of operators. In particular $\left(\frac{d}{dx}\right)^n = \frac{d^n}{dx^n}$. Now, given the definition (1), we can let $e^{t\frac{d}{dt}}$ operate on a function.

$$\left[e^{t\frac{d}{dx}}\right](f) = f + tf' + \frac{t^2}{2}f'' + \cdots + \frac{t^n}{n!}f^{(n)} + \cdots$$

Now, if we plug in $x = 0$, we get

$$f(0) + tf'(0) + \tfrac{t^2}{2}f''(0) + \cdots + \tfrac{t^n}{n!}f^{(n)}(0) + \cdots,$$

which is the Taylor series for $f(t)$, so we can write $\left[e^{t\frac{d}{dx}}\right]$ at $x = 0$ as $f(t)$. We can also write $f(t)$ by using the shift operator, where $\left[E^t f\right](0) = f(t)$. Therefore, $\left[E^t f\right](0) = f(t) = \left[e^{t\frac{d}{dx}} f\right](0)$, and $E^t = e^{t\frac{d}{dx}}$. When we plug $t = 1$, as get $E = e^{\frac{d}{dx}}$, as desired.

1.1 Eigenvectors and eigenvalues of E

If we look at how E operators on sequences, if the sequence $s_0, s_1, s_2, \ldots$ is an eigenvector of E with eigenvalue ϕ, then

$$\begin{aligned} E(s_0, s_1, s_2, \ldots) &= (\phi s_0, \phi s_1, \phi s_2, \ldots) \\ (s_1, s_2, s_3, \ldots) &= (\phi s_0, \phi s_1, \phi s_2, \ldots). \end{aligned}$$

Therefore, $s_{n+1} = \phi s_n$ for all $n \geq 0$, and $s_n = s_0 \phi^n$ for all nonnegative integer n and nonzero s_0.

Also, using methods learned in a differential equations class, we can show that the eigenvectors of $\frac{d}{dx}$ with eigenvalue λ are functions in the form $f(x) = ce^{\lambda x}$ for some constant $c \neq 0$.

These eigenvectors are essentially the same thing as $s_n = s_0 \phi^n = s_0 e^{\lambda n}$, where $\lambda = \ln(\phi)$. Therefore, if s is the sequence $(s_0, s_1, \ldots)$, $Es = \phi s = e^{\lambda} s$ and $\frac{d}{dx} f = \lambda f$.

The operators E and $\frac{d}{dx}$ have the same eigenvectors $ce^{\lambda x}$ but different eigenvalues. We see that $ce^{\lambda x}$ has eigenvalue λ for $\frac{d}{dx}$ and eigenvalue e^{λ} for E.

2 Linear Recurrence Sequences

From previous lectures, we have shown that the following conditions for sequences that satisfy linear recurrences are equivalent. We say that $\{s_n\}_{n \geq 0}$ satisfies a linear recurrence of order k if any of the follow is true:

1. There exists constants $a_0, \ldots, a_{k-1} \in \mathbb{C}$ such that

   $$s_{n+k} = \sum_{i=0}^{k-1} a_i s_{n+i}$$

   for all $n \geq 0$. An example of this is $s_{n+3} = 2s_{n+2} - 5s_{n+1} + s_n$.

2. The terms of the sequences can be expressed as

   $$s_n = \sum_{i=1}^{m} q_i(n) \phi_i^n,$$

   where $\phi_1, \ldots, \phi_m$ are constants in $\mathbb{C}$,$q_1(x), q_2(x), \ldots, q_m(x)$ are polynomials over the complex numbers (in $\mathbb{C}[x]$), and $\sum_{i=1}^{m} \deg(q_i) = k$.

3. The exponential generating function

$$\mathcal{F}(x) = \sum_{n\geq 0} s_n \frac{x^n}{n!}$$

satisfies a linear differential equation of order k. This is true because you shift the series when you differentiate.

We will present a couple more equivalent conditions:

4. The ordinary generating function

$$F_s(x) = \sum_{n\geq 0} s_n x^n$$

is $\frac{P(x)}{Q(x)}$ for some polynomials $P(x), Q(x) \in \mathbb{C}[x]$ such that $\deg(P) < \deg(Q) \leq k$.

5. We can express the terms of the sequence as

$$s_n = v^t A^n w$$

for some k by k matrix $A = (a_{ij})_{i,j=1}^k$ and some vectors v and w.

2.1 Proof of condition 4

We will prove the fourth condition is equivalent to the first condition. If $F_s(x) = \frac{P(x)}{Q(x)}$, where $Q(x) = \sum_{i=0}^{k} a_i x^i$, we get

$$\begin{aligned} F_s(x) &= \frac{P(x)}{Q(x)} \\ Q(x)F_s(x) &= P(x) \\ \left(\sum_{i=0}^{k} a_i x^i\right)\left(\sum_{n\geq 0} s_n x^n\right) &= P(x). \end{aligned}$$

After expanding we get

$$\sum_{m\geq 0}\left(\sum_{i+n=m} a_i s_n\right) x^m = P(x).$$

Now, equate coefficients of x^m. If $m \geq k$, then the coefficient of x^k on the right side is 0, since $P(x)$ must have degree less than k. So if $m \geq k$,

$$\sum_{i+n=m} a_i s_n = 0$$

$$\sum_{i=0}^{k} a_i s_{m-i} = 0. \tag{2}$$

The equation (2) is a linear recurrence of order k. In the equation $F_s(x) = \frac{P(x)}{Q(x)}$, $Q(x)$ encodes the coefficients of the recurrence and $P(x)$ encodes the initial conditions.

2.2 Proof of condition 5

We will prove fifth condition also defines a sequence that satisfies a linear recurrence of order k. To do this, we will use the Cayley Hamilton Theorem:

Theorem 1 *Let A be a square matrix. If $\chi_A(x) = \det(xI - A)$ is the characteristic polynomial of A, then $\chi_A(A) = 0$.*

To show that the fifth definition also defines a sequence that satisfies a linear recurrence of order k, we first show that if the n^{th} term of the sequence can be expressed as $v^t A^n w$ for a k by k matrix A and vectors v and w. Let the characteristic polynomial $\chi_A(x)$ be $\sum_{i=0}^{k} c_i x^i$. Then, for any nonnegative integer n,

$$\begin{aligned}
\sum_{i=0}^{k} c_i s_{n+i} &= \sum_{i=0}^{k} c_i v^t A^{n+i} w \\
&= v^t \left(\sum_{i=0}^{k} c_i A^{n+i} \right) w \\
&= v^t \left[A^n \sum_{i=0}^{k} c_i A^i \right] w \\
&= v^t \left[A^n \chi_A(A) \right] w \\
&= v^t \left[A^n 0 \right] w \\
&= 0.
\end{aligned}$$

One direction is proven. How we need to show that we can represent any linear recurrence in this form.

Given S_n satisfying a linear recurrence

$$\sum_{i=0}^{k} c_i s_{n+i} = 0$$

for all nonnegative integer n and $c_k = 1$, we want to find a matrix A such that its characteristic polynomial is

$$\chi_A(x) = \sum_{i=0}^{k} c_i x^i.$$

One method is to factor χ_A into $\prod_{i=1}^{k} x - \phi_i$, where ϕ_i are the roots of χ_A with multiplicity. Then, let

$$A = \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_k \end{bmatrix}.$$

We could also create an integer matrix if c_i are integers for all $0 \le i \le k$. Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -c_0 & -c_1 & \cdots & -c_{k-2} & c_{k-1} \end{bmatrix},$$

where A has 1's on the superdiagonal, -1 times the coefficients of χ_A in the last row, and 0's everywhere else. Then,

$$\begin{aligned} A \begin{bmatrix} s_n \\ s_{n+1} \\ \vdots \\ s_{n+k-1} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -c_0 & -c_1 & \cdots & -c_{k-2} & c_{k-1} \end{bmatrix} \begin{bmatrix} s_n \\ s_{n+1} \\ \vdots \\ s_{n+k-1} \end{bmatrix} \\ &= \begin{bmatrix} s_{n+1} \\ s_{n+2} \\ \vdots \\ s_{n+k-1} \\ -s_n c_0 - s_{n+1} c_1 + \cdots - s_{n+k-1} c_{k-1} \end{bmatrix}. \end{aligned}$$

Since $\sum_{i=0}^{k} c_i s_{n+i} = 0$, with $c_k = 1$,

$$\begin{bmatrix} s_{n+1} \\ s_{n+2} \\ \vdots \\ s_{n+k-1} \\ -s_n c_0 - s_{n+1} c_1 + \cdots - s_{n+k-1} c_{k-1} \end{bmatrix} = \begin{bmatrix} s_{n+1} \\ s_{n+2} \\ \vdots \\ s_{n+k} \end{bmatrix}.$$

Now, let

$$w = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{k-1} \end{bmatrix}, \; v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then,

$$\begin{aligned} v^t A^n w &= v^t \begin{bmatrix} s_n \\ s_{n+1} \\ \vdots \\ s_{n+k-1} \end{bmatrix} \\ &= s_n. \end{aligned}$$

2.3 Example of a recurrence problem

Example 2 *How many non-self-intersection paths start at the origin in $\mathbb{Z}^2$ with a total of n steps, where all the steps are either up, left, or right?*

One example of such a path with 6 steps starts at (0,0) and traverses (1,0), (2,0), (2,1), (3,1), (3,2), (2,2) in order.

We can encode these paths by words with the letters N, E, and W, where N denotes traveling one unit up, E denotes traveling one unit to the right, and W denotes traveling one unit to the left. The path in the example is given by the word $EENENW$. Since the path must be non-self-intersecting, we are forbidden to have EW or WE as consecutive letters.

Let $f(n)$ be the number of paths with n steps, or the number of words of length n containing the letters N, E, and W without having EW or WE as consecutive letters. If n is at least 2, there are 7 possibilities for the last 2 letters of such a word: EN, WN, NN, EE, NE, WW, or NW. The number of words of length n that end in EN, WN, or NN is $f(n-1)$, since there is no restriction on what can come before N. Similarly, the number of words that can end in NW is $f(n-2)$.

Now we claim that the number of words that can end in WW, EE, or NE is $f(n-1)$. Given any valid word with length $n-1$, it ends with either an E, N or W. If it ends with E, append an E. If it ends with an N, append an E. If it ends in a W, append an W. Therefore, we have found a bijection between the words with length $n-1$ and the words of length n that end with WW, EE, or NE. Therefore, we know $f(n) = 2f(n-1) + f(n-2)$. Solving, we get

$$\begin{aligned} f(n) &= 2f(n-1) + f(n-2) \\ f(n) - 2f(n-1) - f(n-2) &= 0 \\ (E^2 - 2E - 1)f &= 0 \\ (E - (1+\sqrt{2}))(E - (1-\sqrt{2}))f &= 0. \end{aligned}$$

Therefore, $f(n) = a(1+\sqrt{2})^n + b(1-\sqrt{2})^n$ for some constants a and b. We know that $f(0) = 1$, since the word of no letters has length 0, and $f(1) = 3$. Solving for a and b from these initial conditions yields $a = \frac{1+\sqrt{2}}{2}$ and $b = \frac{1-\sqrt{2}}{2}$. Therefore, we have

$$f(n) = \frac{1+\sqrt{2}}{2}(1+\sqrt{2})^n + \frac{1-\sqrt{2}}{2}(1-\sqrt{2})^n,$$

for all nonnegative integers n.