18.312: Algebraic Combinatorics — Lionel Levine

Lecture 9

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1 Order-preserving maps from posets to chains

1.1 Order-preserving maps from n to m

We begin with a question.

Question 1 *How many order-preserving maps are there from the n-chain $\mathbf{n}$ to the m-chain $\mathbf{m}$, $m, n \in \mathbb{N}$? (Equivalently, what is $\mathbf{m}^{\mathbf{n}}$?)*

In an order-preserving map f from $\mathbf{n}$ to $\mathbf{m}$, intervals of $\mathbf{n}$ are mapped to elements of $\mathbf{m}$. Let a_i be the number of $j \in \mathbf{n}$ with $f(j) = i$, $i \in \mathbf{m}$. The set of $\{a_i\}$ uniquely determine f since f preserve order. So we have reduced the question to the following equivalent problem:

Question 2 *How many solutions in non-negative integers $a_1, a_2, \ldots, a_m$ are there of the equation $a_1 + a_2 + \ldots + a_m = n$?*

A solution to this equation is known as a *composition* of n rather than a partition, since the order of the a_i matters. So we would like to know: how many compositions $\alpha(n, m)$ of n are there into m (nonnegative) parts?

We have $\alpha(n,m) = [x^n](1+x^2+x^3+\ldots)^m = \begin{pmatrix} n+m-1 \\ m-1 \end{pmatrix}$. But this means $\alpha(n, m+1) = \begin{pmatrix} n+m \\ m \end{pmatrix} = \begin{pmatrix} n+m \\ n \end{pmatrix} = \alpha(m, n+1)$—a coincidence? No, since we can use a standard bijection here often known as "Stars and Bars" or "Balls and Walls." Each composition of n into m parts is equivalent to placing n stars in a line, and separating them with $m-1$ bars. Hence the number of compositions of n into m parts is $\begin{pmatrix} n+m-1 \\ m-1 \end{pmatrix}$. We can also get an immediate bijection between $\alpha(m, n+1)$ and $\alpha(n, m+1)$ by swapping the stars and bars.

This leads us to ask the following:

Question 3 *Are* $(\mathbf{m}+\mathbf{1})^{\mathbf{n}}$ *and* $(\mathbf{n}+\mathbf{1})^{\mathbf{m}}$ *isomorphic as posets?*

We already showed that the number of maps $\alpha(n, m+1)$ from $\mathbf{n}$ to $\mathbf{m}+\mathbf{1}$ is equal to the number of maps $\alpha(m, n+1)$ from $\mathbf{m}$ to $\mathbf{n}+\mathbf{1}$. But it is also true that $(\mathbf{m}+\mathbf{1})^{\mathbf{n}} \simeq (\mathbf{n}+\mathbf{1})^{\mathbf{m}}$, because

$$(\mathbf{m}+\mathbf{1})^{\mathbf{n}} \simeq (\mathbf{2}^{\mathbf{m}})^{\mathbf{n}} \simeq \mathbf{2}^{\mathbf{m}\times\mathbf{n}} \simeq (\mathbf{2}^{\mathbf{n}})^{\mathbf{m}} \simeq (\mathbf{n}+\mathbf{1})^{\mathbf{m}}.$$

Moreover, all of these are isomorphic to $J(\mathbf{m}\times\mathbf{n})$.

1.2 Order-preserving maps from P to $\mathbf{m}$

Now we consider the more general case of order-preserving maps from a finite poset P to $\mathbf{m}$.

Lemma 4 $\#\{$*Order-preserving maps* $f : P \to \mathbf{m}\} = \#\{$*Multichains* $\hat{0} = I_0 \leq \ldots \leq I_m = \hat{1}$ *in* $J(P)\}$.

Proof: Given $f : P \to \mathbf{m}$, define $I_i = f^{-1}(\{1, 2, \ldots, i\})$. This is an order ideal of P, since if $x \in I_i$ and $y \leq x$, then $f(x) \leq i \implies f(y) \leq f(x) \leq i \implies y \in I_i$. This provides the desired bijection. □

Lemma 5 $\#\{$*Surjective, order-preserving maps* $f : P \to \mathbf{m}\} = \#\{$*Chains* $\hat{0} = I_0 < \ldots < I_m = \hat{1}$ *in* $J(P)\}$.

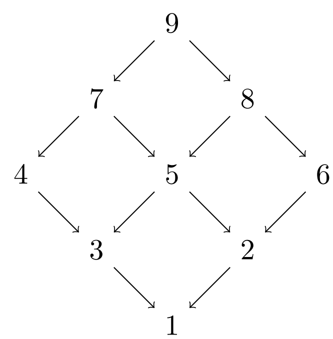
Proof: Again consider I_i as defined above; we only need to show that the inequalities are now strict. But now since f is surjective, there exists x with $f(x) = i+1$, so $x \in I_{i+1}$ but $x \notin I_i$. Hence $I_{i+1} > I_i$. □

2 Linear extensions

Definition 6 *Let P be a poset with $|P| = n$. A linear extension of P is an order-preserving bijection from P to* $\mathbf{n}$.

Alternatively, a linear extension of P is a labeling of the elements of P with a distinct inger from 1 through n, such that a's label is smaller than that of b if $a \leq b$. For example, consider the following Hasse diagram of a linear extension of $P = \mathbf{3} \times \mathbf{3}$:

Figure 1: Example: A linear extension of $\mathbf{3} \times \mathbf{3}$



Define $e(P) = \#\{\text{linear extensions of } P\}$. Then from the section before, we have

$$\begin{aligned} e(P) &= \#\{\text{linear extensions of } P\} \\ &= \#\{\text{surjective, order-preserving maps from } P \to \mathbf{n}\} \\ &= \#\{\text{bijective, order-preserving maps from } P \to \mathbf{n}\} && (\text{since } |P| = n) \\ &= \#\{\text{chains } \hat{0} = I_0 < \ldots < I_m = \hat{1} \text{ in } J(P)\} && (\text{by Lemma 5}) \\ &= \#\{\text{maximal chains in } J(P)\}. && (\text{since } \operatorname{rank}(J(P)) = n) \end{aligned}$$

Next are a few examples.

Example 7 $\#\{$*maximal chains in* $\mathbf{m} \times \mathbf{n}\}$.

We have

$$\begin{aligned} & \mathbf{m} \times \mathbf{n} = \mathbf{2}^{\mathbf{m-1}} \times \mathbf{2}^{\mathbf{n-1}} = \mathbf{2}^{\mathbf{m-1+n-1}} \\ \implies & \mathbf{m} \times \mathbf{n} = J((\mathbf{m-1}) + (\mathbf{n-1})), \end{aligned}$$

so

$$\begin{aligned}\#\{\text{maximal chains in } \mathbf{m}\times\mathbf{n}\} &= \#\{\text{linear extensions of } (\mathbf{m-1})+(\mathbf{n-1})\}\\ &= e((\mathbf{m-1})+(\mathbf{n-1}))\\ &= \binom{m+n-2}{m-1}.\end{aligned}$$

Example 8 *#{maximal chains in boolean algebra B_n}.*

Since $B_n = J(\underbrace{1+1+\ldots+1}_{n})$, the number of maximal chains in B_n is $e(1+1+\ldots+1) = n!$.

3 Incidence algebras

3.1 Definitions

Let P be a finite poset and K a finite field. We will usually take $K = \mathbb{C}$.

Definition 9 *$Int(P) = \{intervals\ [x, y] \subseteq P,\ x \le y\}$. (The empty set is not an interval.)*

Definition 10 *The incidence algebra $I(P)$ of a poset P is the vector space of all functions $f : Int(P) \to K$.*

$I(P)$ has multiplication $(fg)[x, y] = \sum_{x\le z\le y} f([x, z])g([z, y])$.

An equivalent (really the dual) definition of $I(P)$ is the following:

Definition 11 *$I(P)$ is the set of formal linear combinations of intervals $\sum_{[x,y]\in Int(P)} f([x, y])[x, y]$.*

with multiplication

$$[x, y][z, w] = \begin{cases}[x, w], & y = z\\ 0, & \text{otherwise.}\end{cases} \tag{1}$$

We can check that

$$\begin{aligned}
&\left(\sum_{[x,y]\in \mathrm{Int}(P)} f([x,y])[x,y]\right)\left(\sum_{[z,w]\in \mathrm{Int}(P)} g([z,w])[z,w]\right) \\
&= \sum_{[x,y],[z,w]\in \mathrm{Int}(P)} f([x,y])g([z,w])\underbrace{[x,y][z,w]}_{0 \text{ unless } y=z} \\
&= \sum_{x\le y\le w} f([x,y])g([y,w])[x,w] \\
&= \sum_{[x,w]\in \mathrm{Int}(P)} f([x,y])g([y,z])[x,w].
\end{aligned}$$

Another equivalent definition involves matrices. Let the elements of P be $\{x_1, \ldots, x_n\}$. Then:

Definition 12 $I(P) = \{n \times n$ *matrices* $A \mid a_{ij} \in K, a_{ij} = 0$ *unless* $x_i \le x_j\}$.

So each interval $[x_i, x_j]$ with $x_i \le x_j$ is represented as the matrix e_{ij} with just one nonzero entry a_{ij}.

Example 13 $P = \mathbf{n}$. *Then* $I(P)$ *is the set of upper triangular* $n \times n$ *matrices.*

Example 14 $P = B_2$. *Then* $I(P)$ *looks like* $\begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$, *where * denotes a nonzero element of* K.

3.2 ζ and 1; more chain-counting

Two important elements of $I(P)$ are the zeta element ζ and the identity, 1.

Definition 15 $\zeta([x,y]) = 1 \forall [x,y] \in Int(P)$.

Example 16

$$\zeta^2([x,y]) = \sum_{x\le z\le y} \zeta([x,z])\zeta([z,y]) = \#\{[x,y] \in Int(P)\}.$$

Example 17

$$\zeta^k([x,y]) = \sum_{x=z_0\leq\ldots\leq z_k=y}\prod_{i=1}^{k}\zeta([z_{i-1},z_i]) = \sum_{x=z_0\leq\ldots\leq z_k=y} 1$$

, so $\zeta^k([x,y]) = \#\{$*multichains* $x = z_0 \leq z_1 \leq \ldots \leq z_k = y.\}$

Definition 18 $1([x,y]) = \delta_{xy} = \begin{cases} 1, & x = y, \\ 0, & \textit{otherwise.} \end{cases}$

Now consider $(\zeta - 1) \in I(P)$; we have $(\zeta-1)([x,y]) = \begin{cases} 0, & x = y, \\ 1, x \neq y. \end{cases}$. So

$$(\zeta-1)^k([x,y]) = \sum_{x=z_0\leq\ldots\leq z_k=y}(\zeta-1)([z_{i-1},z_i])$$

which counts precisely the number of chains $x = z_0 < z_1 < \ldots < z_k = y$.

Using this result, we have the following:

Lemma 19 $(2 - \zeta)$ *is invertible in* $I(P)$, *and* $(2-\zeta)^{-1}([x,y]) = \#\{$*chains from* x *to* y*, regardless of length.*$\}$

Proof: We use the matrix definition of $I(P)$; we have $f([x,x]) \neq 0 \forall x \in P \Leftrightarrow f$ is invertible. So since $(2-\zeta)([x,y]) = \begin{cases} 1, & x = y, \\ -1, & x < y \end{cases}$, $2-\zeta$ is invertible. Now let $r = \text{rank } P$; since $(\zeta-1)^k$ counts the number of chains of length $k+1$, we must have $(\zeta-1)^r = 0$. But

$$\left(1 + (\zeta-1) + (\zeta-1)^2 + \ldots + (\zeta-1)^{r-1}\right)(1 - (1-\zeta)) = 1 - (\zeta-1)^r = 1,$$

so the inverse of $2-\zeta$ is $1 + (\zeta-1) + (\zeta-1)^2 + \ldots + (\zeta-1)^{r-1}$, which (when applied to the interval $[x,y]$) is the total number of chains of any length from x to y. ■ □

3.3 For next time...

Claim. $f^{-1}([x,y])$ depends only on the poset structure of $[x,y]$.

Definition 20 *The Möbius element* $\mu \in I(P)$ *is defined by* $\mu = \zeta^{-1}$.