1 Order-preserving maps from posets to chains

1.1 Order-preserving maps from $n$ to $m$

We begin with a question.

**Question 1** How many order-preserving maps are there from the $n$-chain $n$ to the $m$-chain $m$, $m, n \in \mathbb{N}$? (Equivalently, what is $m^n$?)

In an order-preserving map $f$ from $n$ to $m$, intervals of $n$ are mapped to elements of $m$. Let $a_i$ be the number of $j \in n$ with $f(j) = i$, $i \in m$. The set of $\{a_i\}$ uniquely determine $f$ since $f$ preserve order. So we have reduced the question to the following equivalent problem:

**Question 2** How many solutions in non-negative integers $a_1, a_2, \ldots, a_m$ are there of the equation $a_1 + a_2 + \ldots + a_m = n$?

A solution to this equation is known as a composition of $n$ rather than a partition, since the order of the $a_i$ matters. So we would like to know: how many compositions $\alpha(n,m)$ of $n$ are there into $m$ (nonnegative) parts?

We have $\alpha(n,m) = [x^n](1 + x^2 + x^3 + \ldots)^m = \binom{n+m-1}{m-1}$. But this means $\alpha(n,m+1) = \binom{n+m}{m} = \binom{n+m}{n} = \alpha(m,n+1)$—a coincidence? No, since we can use a standard bijection here often known as “Stars and Bars” or “Balls and Walls.” Each composition of $n$ into $m$ parts is equivalent to placing $n$ stars in a line, and separating them with $m-1$ bars. Hence the number of compositions of $n$ into $m$ parts is $\binom{n+m-1}{m-1}$. We can also get an immediate bijection between $\alpha(m,n+1)$ and $\alpha(n,m+1)$ by swapping the stars and bars.

This leads us to ask the following:
Question 3 Are \((m+1)^n\) and \((n+1)^m\) isomorphic as posets?

We already showed that the number of maps \(\alpha(n, m+1)\) from \(n\) to \(m+1\) is equal to the number of maps \(\alpha(m, n+1)\) from \(m\) to \(n+1\). But it is also true that \((m+1)^n \simeq (n+1)^m\), because

\[
(m+1)^n \simeq (2^m)^n \simeq 2^{m\times n} \simeq (2^n)^m \simeq (n+1)^m.
\]

Moreover, all of these are isomorphic to \(J(m \times n)\).

1.2 Order-preserving maps from \(P\) to \(m\)

Now we consider the more general case of order-preserving maps from a finite poset \(P\) to \(m\).

**Lemma 4** \#\{Order-preserving maps \(f : P \rightarrow m\}\} = \#\{Multichains \(\hat{0} = I_0 \leq \ldots \leq I_m = \hat{1}\) in \(J(P)\}\).

**Proof:** Given \(f : P \rightarrow m\), define \(I_i = f^{-1}(\{1, 2, \ldots, i\})\). This is an order ideal of \(P\), since if \(x \in I_i\) and \(y \leq x\), then \(f(x) \leq i \implies f(y) \leq f(x) \leq i \implies y \in I_i\). This provides the desired bijection. \(\square\)

**Lemma 5** \#\{Surjective, order-preserving maps \(f : P \rightarrow m\}\} = \#\{Chains \(\hat{0} = I_0 < \ldots < I_m = \hat{1}\) in \(J(P)\)\}.

**Proof:** Again consider \(I_i\) as defined above; we only need to show that the inequalities are now strict. But now since \(f\) is surjective, there exists \(x\) with \(f(x) = i + 1\), so \(x \in I_{i+1}\) but \(x \notin I_i\). Hence \(I_{i+1} > I_i\). \(\square\)
2 Linear extensions

**Definition 6** Let $P$ be a poset with $|P| = n$. A linear extension of $P$ is an order-preserving bijection from $P$ to $n$.

Alternatively, a linear extension of $P$ is a labeling of the elements of $P$ with a distinct integer from 1 through $n$, such that $a$’s label is smaller than that of $b$ if $a \leq b$. For example, consider the following Hasse diagram of a linear extension of $P = 3 \times 3$:

![Figure 1: Example: A linear extension of $3 \times 3$](image)

Define $e(P) = \#\{\text{linear extensions of } P\}$. Then from the section before, we have

\[
e(P) = \#\{\text{linear extensions of } P\} \\
= \#\{\text{surjective, order-preserving maps from } P \rightarrow n\} \\
= \#\{\text{bijective, order-preserving maps from } P \rightarrow n\} \quad \text{(since } |P| = n) \\
= \#\{\text{chains } \hat{0} = I_0 < \ldots < I_m = \hat{1} \text{ in } J(P)\} \quad \text{(by Lemma 5)} \\
= \#\{\text{maximal chains in } J(P)\} \quad \text{(since rank}(J(P)) = n) \\
\]

Next are a few examples.

**Example 7** $\#\{\text{maximal chains in } m \times n\}$.

We have

\[
m \times n = 2^{m-1} \times 2^{n-1} = 2^{m+n-1} \\
\implies m \times n = J((m - 1) + (n - 1)),
\]
\[
\#\{\text{maximal chains in } m \times n\} = \#\{\text{linear extensions of } (m - 1) + (n - 1)\} \\
= e((m - 1) + (n - 1)) \\
= \binom{m + n - 2}{m - 1}.
\]

**Example 8** \#\{maximal chains in boolean algebra \(B_n\}\).

Since \(B_n = J(1 + 1 + \ldots + 1)\), the number of maximal chains in \(B_n\) is \(e(1+1+\ldots+1) = n!\).

### 3 Incidence algebras

#### 3.1 Definitions

Let \(P\) be a finite poset and \(K\) a finite field. We will usually take \(K = \mathbb{C}\).

**Definition 9** \(Int(P) = \{\text{intervals } [x, y] \subseteq P, x \leq y\}\). (*The empty set is not an interval.*)

**Definition 10** The incidence algebra \(I(P)\) of a poset \(P\) is the vector space of all functions \(f : Int(P) \to K\).

\(I(P)\) has multiplication \((fg)[x, y] = \sum_{x \leq z \leq y} f([x, z])g([z, y])\).

An equivalent (really the dual) definition of \(I(P)\) is the following:

**Definition 11** \(I(P)\) is the set of formal linear combinations of intervals \(\sum_{[x, y] \in Int(P)} f([x, y])[x, y]\). with multiplication

\[
[x, y][z, w] = \begin{cases} 
[x, w], & y = z \\
0, & \text{otherwise.} 
\end{cases}
\] (1)
We can check that
\[
\left( \sum_{[x,y] \in \text{Int}(P)} f([x,y])[x,y] \right) \left( \sum_{[z,w] \in \text{Int}(P)} g([z,w])[z,w] \right) \\
= \sum_{[x,y],[z,w] \in \text{Int}(P)} f([x,y])g([z,w])[x,y][z,w] \\
= \sum_{x \leq y \leq w} f([x,y])g([y,w])[x,w] \\
= \sum_{[x,w] \in \text{Int}(P)} f([x,y])g([y,z])[x,w].
\]

Another equivalent definition involves matrices. Let the elements of \( P \) be \( \{x_1, \ldots, x_n\} \). Then:

**Definition 12** \( I(P) = \{n \times n \text{ matrices } A \mid a_{ij} \in K, a_{ij} = 0 \text{ unless } x_i \leq x_j \} \).

So each interval \([x_i, x_j]\) with \( x_i \leq x_j \) is represented as the matrix \( e_{ij} \) with just one nonzero entry \( a_{ij} \).

**Example 13** \( P = n \). Then \( I(P) \) is the set of upper triangular \( n \times n \) matrices.

**Example 14** \( P = B_2 \). Then \( I(P) \) looks like
\[
\begin{pmatrix}
* & * & * & * \\
0 & * & * & * \\
0 & 0 & * & * \\
0 & 0 & 0 & *
\end{pmatrix},
\]
where * denotes a nonzero element of \( K \).

### 3.2 \( \zeta \) and 1; more chain-counting

Two important elements of \( I(P) \) are the zeta element \( \zeta \) and the identity, 1.

**Definition 15** \( \zeta([x,y]) = 1 \forall [x,y] \in \text{Int}(P) \).

**Example 16**
\[
\zeta^2([x,y]) = \sum_{x \leq z \leq y} \zeta([x,z])\zeta([z,y]) = \#\{[x,y] \in \text{Int}(P)\}.
\]
Example 17

\[ \zeta^k([x, y]) = \sum_{x=z_0 \leq \ldots \leq z_k = y} \prod_{i=1}^{k} \zeta([z_{i-1}, z_i]) = \sum_{x=z_0 \leq \ldots \leq z_k = y} 1 \]

, so \( \zeta^k([x, y]) = \#\{\text{multichains } x = z_0 \leq z_1 \leq \ldots \leq z_k = y\} \)

Definition 18 \( \delta_{xy} = \begin{cases} 1, & x = y, \\ 0, & \text{otherwise}. \end{cases} \)

Now consider \((\zeta - 1) \in I(P)\); we have \((\zeta - 1)([x, y]) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases} \) So

\[ (\zeta - 1)^k([x, y]) = \sum_{x=z_0 \leq \ldots \leq z_k = y} (\zeta - 1([z_{i-1}, z_i])) \]

which counts precisely the number of chains \( x = z_0 < z_1 < \ldots < z_k = y \).

Using this result, we have the following:

Lemma 19 \((2 - \zeta)\) is invertible in \( I(P) \), and \( (2 - \zeta)^{-1}([x, y]) = \#\{\text{chains from } x \text{ to } y, \text{ regardless of length}\} \)

Proof: We use the matrix definition of \( I(P) \); we have \( f([x, x]) \neq 0 \forall x \in P \iff f \text{ is invertible.} \)

So since \((2 - \zeta)([x, y]) = \begin{cases} 1, & x = y, \\ -1, & x < y. \end{cases} \), \( 2 - \zeta \) is invertible. Now let \( r = \text{rank } P \); since \((\zeta - 1)^k\) counts the number of chains of length \( k + 1 \), we must have \((\zeta - 1)^r = 0\). But

\[ (1 + (\zeta - 1) + (\zeta - 1)^2 + \ldots + (\zeta - 1)^{r-1}) (1 - (1 - \zeta)) = 1 - (\zeta - 1)^r = 1, \]

so the inverse of \( 2 - \zeta \) is \( 1 + (\zeta - 1) + (\zeta - 1)^2 + \ldots + (\zeta - 1)^{r-1} \), which (when applied to the interval \([x, y]\)) is the total number of chains of any length from \( x \) to \( y \).

3.3 For next time...

Claim. \( f^{-1}([x, y]) \) depends only on the poset structure of \([x, y]\).

Definition 20 The M"obius element \( \mu \in I(P) \) is defined by \( \mu = \zeta^{-1} \).