18.312: Algebraic Combinatorics — Lionel Levine

Practice Problems for Midterm

The questions on the midterm on Thursday March 10 will be similar to these, but there will be fewer of them. Explain your reasoning to receive full credit, even for computational questions. If a result was proved in lecture or on the problem sets, you may use it without reproducing the proof.

PM1 The *Lucas numbers* are defined by the recurrence

$$L_{n+2} = L_{n+1} + L_n, \qquad n \geq 1$$

with initial conditions $L_1 = 1$ and $L_2 = 3$.

(a) Show that L_n is the number of circular necklaces $(b_1, \ldots, b_n)$ with beads $b_i \in \{red, blue\}$ such that no two consecutive beads are red.

(b) Show that $L_p \equiv L_1 \pmod{p}$ for all primes p.

(c) Let $n \in \mathbb{N}$. Generalize part (b) by finding for a linear combination of the numbers $L_1, \ldots, L_n$ with coefficients ± 1 that is divisible by n.

. . . . . .

PM2 How many of the numbers $1, \ldots, 10! - 1$ are relatively prime to $10!$?

. . . . . .

PM3 Suppose that f is a function on $\mathbb{N}$ satisfying

$$\sum_{d|n} \frac{f(d)}{d} = \frac{1}{n} \qquad \text{for all } n \geq 1.$$

Find $f(60)$.

$$\cdots\cdots$$

PM4 Evaluate $\sum_{d|n} \frac{\mu(d)}{d}$.

$$\cdots\cdots$$

PM5 Evaluate $\sum_{n\geq 1} \frac{\mu(n)}{2^n-1} = 1 - \frac{1}{3} - \frac{1}{7} - \frac{1}{31} + \frac{1}{63} - \frac{1}{127} + \frac{1}{1023} - \frac{1}{2047} - \ldots.$

(Hint: expand $\sum_{n\geq 1} \frac{\mu(n)x^n}{1-x^n}$ as a power series that converges for $|x| < 1$.)

$$\cdots\cdots$$

PM6 Prove that if p is prime, then $p|S(p,k)$ for all $1 < k < p$.

$$\cdots\cdots$$

PM7

Let $\{a_n\}_{n\geq 0}$ be the sequence defined by

$$\frac{1}{1-x+x^2} = \sum_{n\geq 0} a_n x^n, \qquad |x| < 1.$$

(a) Find a linear recurrence satisfied by a_n.

(b) Prove that a_n is periodic with period 6.

(c) Find a linear recurrence satisfied by the sequence $b_n = a_n + 5^n$.

(d) Find a linear recurrence satisfied by the sequence $c_n = na_n$.

(e) Find a linear recurrence satisfied by the sequence $d_n = a_{2n}$.

$$\cdots\cdots$$

PM8 Find a matrix M and vectors v, w such that $v^T M^n w = n^2$ for all $n \geq 0$.

$\cdots\cdots$

PM9 Let P and Q be posets. Show that $J(P \oplus \mathbf{1} \oplus Q) \simeq J(P) \oplus J(Q)$.

$\cdots\cdots$

PM10 Let P be the lattice of subgroups of S_3.

(a) Draw the Hasse diagrams of P and $J(P)$.

(b) How many maximal chains does $J(P)$ have?

$\cdots\cdots$

M11 Consider the $n \times n$ matrix

$$M = \begin{pmatrix} 1 & -1 & -1 & \dots & -1 & -1 \\ 0 & 1 & -1 & \dots & -1 & -1 \\ 0 & 0 & 1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

(a) Express M in terms of the ζ element of an incidence algebra $I(P)$ for some poset P.

(b) Use part (a) to give a formula for the entries of M^{-1}.