P4 Express in terms of the Riemann zeta function $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$:

(a) $\sum_{n \geq 1} \frac{\sigma(n)}{n^s}$, where $\sigma(n) = \sum_{d|n} d$.

(b) $\sum_{n \geq 1} \frac{\phi(n)}{n^s}$.

(c) $\sum_{n \geq 1} \frac{g(n)}{n^s}$, where $g(n) = \sum_{k=1}^{n} \gcd(k, n)$.

P5 Find the coefficient of $n^{-s}$ in $\zeta(s)/\zeta(2s)$.

P6 Let $\tau(n)$ be the number of divisors of $n$. Prove that

$$\sum_{d|n} \phi(d) \sigma\left(\frac{n}{d}\right) = n \tau(n).$$

P7 Give a combinatorial proof of the identity

$$\sum_{k=1}^{n} c(n, k)x^k = x(x + 1) \cdots (x + n - 1)$$

by counting pairs $(\pi, f)$ where $\pi \in S_n$ and $f : \{\text{cycles of } \pi\} \to [x]$ in two ways.
Compute the inverse of the matrix

\[
M_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 \\
1 & 4 & 6 & 4 & 1
\end{pmatrix}.
\]

Make a conjecture about the inverse of \( M_n = \left( \binom{i}{j} \right)_{i,j=0}^n \). Prove your conjecture by expressing \( M_n \) as a change of basis matrix.