18.312: Algebraic Combinatorics — Lionel Levine

Problem Set 2

Due in class on Feb 17, 2011

P4 Express in terms of the Riemann zeta function $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$:

(a) $\displaystyle\sum_{n\geq 1} \frac{\sigma(n)}{n^s},$ where $\sigma(n) = \displaystyle\sum_{d|n} d.$

(b) $\displaystyle\sum_{n\geq 1} \frac{\phi(n)}{n^s}.$

(c) $\displaystyle\sum_{n\geq 1} \frac{g(n)}{n^s},$ where $g(n) = \displaystyle\sum_{k=1}^{n} \gcd(k,n).$

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P5 Find the coefficient of n^{-s} in $\zeta(s)/\zeta(2s)$.

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P6 Let $\tau(n)$ be the number of divisors of n. Prove that

$$\sum_{d|n} \phi(d)\sigma(\frac{n}{d}) = n\tau(n).$$

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P7 Give a combinatorial proof of the identity

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1)$$

by counting pairs (π, f) where $\pi \in S_n$ and $f : \{\text{cycles of } \pi\} \to [x]$ in two ways.

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P8 Compute the inverse of the matrix

$$M_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

Make a conjecture about the inverse of $M_n = (\binom{i}{j})_{i,j=0}^n$. Prove your conjecture by expressing M_n as a change of basis matrix.