18.312: Algebraic Combinatorics | Lionel Levine

Problem Set 3

Due at the beginning of class on Feb 24, 2011

P9 Find a linear recurrence satisfied by the sequence $\{S(n,3)\}_{n\geq 3}$.

$\cdots\cdots$

P10 Give an example (with proof) of a sequence that does not satisfy any linear recurrence.

$\cdots\cdots$

P11 Let $k \geq 1$ be an integer, and let $\omega = e^{2\pi i/k}$. Prove that any sequence of complex numbers $a_0, a_1, a_2, \ldots$ that is periodic with period k (that is, $a_{n+k} = a_n$ for all $n \geq 0$) can be written in the form

$$a_n = c_0 + c_1\omega^n + c_2\omega^{2n} + \ldots + c_{k-1}\omega^{(k-1)n}$$

for some constants $c_0, \ldots, c_{k-1} \in \mathbb{C}$.

$\cdots\cdots$

P12 Let $f(n)$ be the number of ways to make n cents out of pennies, nickels, dimes and quarters. For example, $f(10) = 4$ (10 pennies, or 5 pennies and 1 nickel, or 2 nickels, or 1 dime).

(a) Show that $\displaystyle\sum_{n\geq 0} f(n)x^n = \frac{1}{1-x}\frac{1}{1-x^5}\frac{1}{1-x^{10}}\frac{1}{1-x^{25}}$.

(b) Write down a linear recurrence that $f(n)$ satisfies.

(c) Prove that $f(50n) = an^3 + bn^2 + cn + 1$ for some constants a, b, c.

$\cdots\cdots$

P13 If a_n satisfies a linear recurrence of order k, and b_n satisfies a linear recurrence of order ℓ, show that

- $a_n + b_n$ satisfies a linear recurrence of order $k + \ell$.
- $a_n b_n$ satisfies a linear recurrence of order $k\ell$.
- a_n^2 satisfies a linear recurrence of order $\binom{k+1}{2}$.

. . . . . .

P14 Find the values of the infinite series

$$\sum_{n=1}^{\infty} \frac{F_n}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{5}{32} + \frac{8}{64} + \frac{13}{128} + \dots$$

and

$$\sum_{n=1}^{\infty} \frac{F_n}{n!} = 1 + \frac{1}{2} + \frac{2}{6} + \frac{3}{24} + \frac{5}{120} + \frac{8}{720} + \frac{13}{5040} + \dots.$$

. . . . . .