18.312: Algebraic Combinatorics — Lionel Levine

Problem Set 4

Due at the beginning of class on Tuesday March 8, 2011

P15 Let P, Q, R be finite posets. Prove that $P^{Q+R} \simeq P^Q \times P^R$ and $(P^Q)^R \simeq P^{Q\times R}$.

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P16 Let P be a finite poset. An *upper order ideal* is a subset $U \subset P$ such that if $x \leq y$ and $x \in U$, then $y \in U$. An *antichain* is a subset $A \subset P$ such that no two elements of A are comparable (that is, if $x, y \in A$ are distinct then $x \not\leq y$ and $y \not\leq x$). Show that the number of order ideals, upper order ideals, and antichains of P are all equal.

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P17 Let D_n be the set of positive divisors of n, partially ordered by divisibility.

(a) Show that D_n is a lattice. Is it distributive? If so, describe the poset P_n such that $D_n = J(P_n)$.

(b) Show that $D_{mn} \simeq D_m \times D_n$ If m and n are relatively prime.

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P18 Let $s = (s_0, s_1, s_2, \ldots)$ be a sequence with $s_0 = 1$ satisfying both of the following linear recurrences:

$$s_{n+3} - 3s_{n+2} + s_{n+1} + 2s_n = 0$$

$$s_{n+3} - 3s_{n+1} - 2s_n = 0.$$

Find s_{100}.

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P19 Let $\{s_{m,n}\}_{m,n\geq 0}$ be an infinite two-dimensional array, $s_{m,n} \in \mathbb{C}$, such that each column obeys a linear recurrence

$$\sum_{i=0}^{k} a_i s_{m+i,n} = 0, \qquad m, n \geq 0$$

and each row obeys a linear recurrence

$$\sum_{j=0}^{\ell} b_j s_{m,n+j} = 0, \qquad m, n \geq 0.$$

Prove that the sequence $\{s_{n,n}\}_{n\geq 0}$ obeys a linear recurrence of order $k\ell$.

(Hint: consider horizontal and vertical shift operators E and F, and prove that the vector space spanned by the arrays $E^i F^j s$ is finite dimensional.)