18.312: Algebraic Combinatorics — Lionel Levine

Problem Set 7

Due at the beginning of class on Thursday April 21, 2011

P30

(a) Find the number of domino tilings of a $2 \times n$ rectangle.

(In class we found a complicated formula for the number of domino tilings of an $m \times n$ rectangle when mn is even, but there is a simpler answer when $m = 2$.)

(b) Fix $m \in \mathbb{N}$ and let t_n be the number of domino tilings of an $m \times n$ rectangle. Show that the sequence $\{t_n\}_{n \geq 1}$ obeys a linear recurrence of order at most 2^m.

(Hint: For each subset $S \subseteq [m]$, consider the number $t_{S,n}$ of domino tilings of the region $([m] \times [n]) \cup \{(s, n+1) \mid s \in S\}$. Find a $2^m \times 2^m$ matrix M such that the vector $v_n := (t_{S,n})_{S \subseteq [m]}$ equals $M v_{n-1}$.)

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P31 The *honeycomb lattice* is the infinite graph with vertices

$$V = \{m + n\omega \,:\, m, n \in \mathbb{Z},\ m + n \text{ is not divisible by } 3\}$$

where $\omega = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of 1; and edges

$$E = \{(u, v) \in V \times V \,:\, |u - v| = 1\}.$$

Let G be a finite induced subgraph of the honeycomb lattice. Prove that the number of perfect matchings of G equals $\sqrt{|\det K|}$, where K is the adjacency matrix

$$K_{uv} = \begin{cases} 1 & \text{if } (u, v) \in E \\ 0 & \text{else.} \end{cases}$$

In other words, Kasteleyn's theorem holds without weights!

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P32 Let G be a connected graph on n vertices, and let T be a spanning subgraph of G. Prove that any two of the conditions below imply the third.

(1) T is connected.

(2) T is a forest.

(3) T has exactly $n-1$ edges.