CoEulerian Graphs

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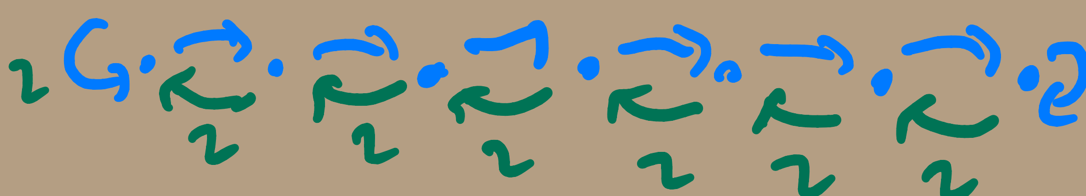
joint work with Matt Farrell



CMU ACO Seminar

4/16/2020

$G = (V, E)$ FINITE DIRECTED GRAPH

ASSUME STRONGLY CONNECTED: $\forall x, y \in V$

$\exists$ PATHS $x \rightsquigarrow y$, $y \rightsquigarrow x$.

__DEF__ G IS __EULERIAN__ IF $\exists$ TOUR

$$\begin{tikzcd} {} \arrow[r, "e_0"] & {} \arrow[r, "e_1"] & \cdots \arrow[r, "e_{m-1}"] & {} \arrow[lll, bend left, "e_m"] \end{tikzcd}$$

SUCH THAT EACH DIRECTED EDGE OF G APPEARS EXACTLY ONCE.

__THM__ G EULERIAN $\iff$ $\mathrm{indeg}(x) = \mathrm{outdeg}(x)$ $\forall x \in V$.

A DIRECTED GRAPH LAPLACIAN $\Delta : \mathbb{Z}^V \to \mathbb{Z}^V$

$$\Delta f(v) = \underbrace{d_v}_{\text{out}} f(v) - \sum_{\underbrace{e : e^+ = v}_{\text{incoming edges}}} f(e^-)$$

$$d_v = \text{outdeg}(v) = \#\{e \in E \mid e^- = v\}$$

e is oriented from e^- to e^+

$$\Delta = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} - \begin{pmatrix} a_{ij} \end{pmatrix}$$

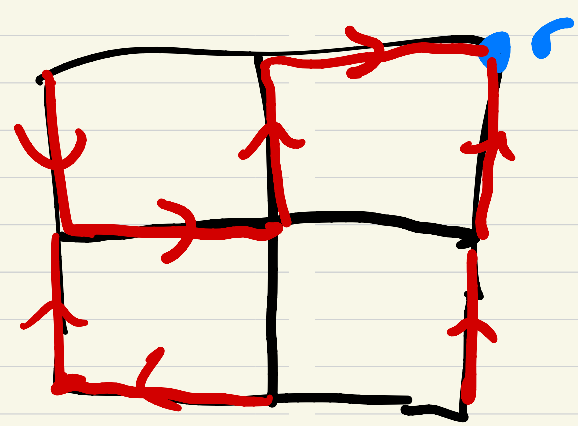
DEF AN ORIENTED SPANNING TREE OF G

ORIENTED TOWARD $r \in V$ IS

AN ACYCLIC SUBGRAPH $T = (V, A)$ SATISFYING

$$\text{outdeg}_T(r) = 0$$

$$\text{outdeg}_T(v) = 1 \qquad \forall v \in V - \{r\}$$



BIDIRECTED

MTT, MCTT, BEST : 3 CORNERSTONES OF "ALG. DIR. GRAPH THY"

MATRIX-TREE THEOREM: LET $\kappa(r)$ BE THE NUMBER OF ORIENTED SP. TREES OF G ORIENTED TOWARD r. THEN

$$\kappa(r) = \det(\Delta_r)$$

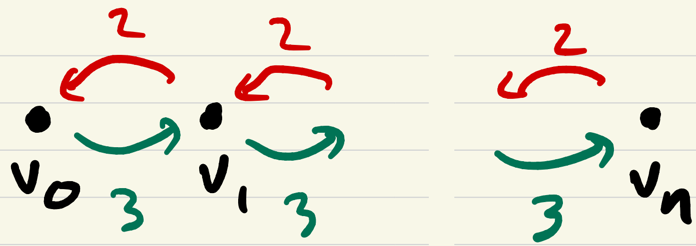
CROSS OFF ROW & COL r FROM Δ.

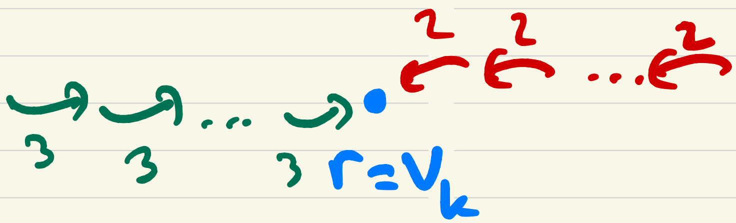
Pham 2016: $\phi = \text{g.c.d.}\{\kappa(r)\}_{r \in V}$.

MEASURES "EULERIANNESS" OF G.

$\phi = \kappa(r) \ \forall r \iff G$ IS EULERIAN.

DEF: G IS COEULERIAN IF $\phi = 1$.





$$\chi(r) = 3^k 2^{n-k}$$
$$\phi = \gcd(3^n, 2^n) = 1.$$

DEF: CHIP CONFIG $\sigma : V \to \mathbb{Z}$ "$\sigma(v)$ CHIPS AT VERTEX v"

FIRING VERTEX v: SEND 1 CHIP FROM v ALONG EACH EDGE WITH $e^- = v$.

LEGAL IF $\sigma(v) \geq d_v$

DEF σ STABILIZES IF $\exists$ LEGAL FIRING SEQ. $\sigma, \sigma_1, \sigma_2, \ldots, \sigma_k$ WITH σ_k STABLE: $\sigma_k(v) < d_v \ \forall v$.

HALTING PROBLEM FOR CHIP-FIRING:

GIVEN (the adj matrix of) G, CHIP CONFIG $\sigma \in \mathbb{Z}^V$

DECIDE WHETHER σ STABILIZES.

SOMETIMES EASY: IF $|\sigma| > |\sigma_{max}| = \sum_{v \in V} (d_v - 1)$,

BY PIGEONHOLE THERE IS ALWAYS AN UNSTABLE VERTEX

$\sigma_1, \dots, \sigma_k$

$|\sigma_k| = |\sigma| \implies \exists v \quad \sigma_k(v) \geq d_v.$

CONVERSE? USUALLY FAILS, BUT...

THM:

(Classical) Eulerian	CoEulerian
$\ker(\Delta) = \mathbb{Z}1$	$\mathrm{Im}(\Delta) = \mathbb{Z}_0^V$
$\Updownarrow$	$\Updownarrow$
$\phi = \kappa(s) \quad \forall s \in V$	$\phi = 1$
$\Updownarrow$	$\Updownarrow$
G has Eul. tour	$\boxed{\forall \sigma \in \mathbb{Z}^V \text{ with } \lvert\sigma\rvert \leq \lvert\sigma_{max}\rvert, \ \sigma \text{ stabilizes!}}$

G. TARDOS 1988: G BIDIRECTED, IF $\exists$ LEGAL FIRING SEQ. FOR σ WHERE EVERY $v \in V$ FIRES AT LEAST ONCE, THEN σ DOES NOT STABILIZE.

$\rightsquigarrow$ POLY. TIME ALGO TO CHECK FOR STABILIZATION.

BJÖRNER–LOVÁSZ 1992: $\exists!$ PRIMITIVE $\pi \in \mathbb{N}^V$ SUCH THAT $\Delta\pi = 0$. IF EACH $v \in V$ FIRES $\geq \pi(v)$ TIMES, THEN σ DOES NOT STABILIZE.

BUT $|\pi|$ CAN BE EXPONENTIALLY LARGE!

MARKOV CHAIN TREE THM: $\boxed{\pi = \frac{1}{\phi} \kappa}$

THM (FARRELL–L.) THE HALTING PROBLEM FOR CHIP-FIRING IS NP-COMPLETE FOR S.C. MULTIGRAPHS

OPEN: IS IT STILL NP-COMPLETE FOR SIMPLE DIRECTED GRAPHS?

THM (Nguyen–Wood 2018):

$$P\left(\vec{G}(n,p) \text{ IS COEULERIAN}\right) \longrightarrow \prod_{k\geq 2} \frac{1}{\zeta(k)} \approx 0.436\ldots$$

AS $n \to \infty$

LIMIT DOESN'T DEPEND ON p.

FIX q PRIME

$$P(\Delta : \mathbb{Z}_q^V \to \mathbb{Z}_q^V \text{ ONTO})$$