1. Let X_n be a sequence of random variables with $EX_n \to \infty$. We say that X_n is *concentrated around its mean* (or just concentrated, for short) if

$$\frac{X_n}{EX_n} \to 1 \qquad \text{in probability.}$$

For example, in class we showed that the time it takes the coupon collector to collect all n coupons is concentrated.

   (a) Show that if $(\operatorname{Var} X_n)/(EX_n)^2 \to 0$ then X_n is concentrated. Does the converse hold?

   (b) Which of the following are concentrated? Prove your answers!

      (i) $B_n \sim \text{Bin}(n, 1/2)$ (binomial distribution)

      (ii) $U_n \sim \text{Unif}(0, n)$ (uniform distirbution on the interval $(0, n)$)

      (iii) $G_n \sim \text{Geom}(1/n)$ (geometric distribution with mean n)

2. Let $X_1, X_2, \ldots$ be independent Unif$(0, 1)$ random variables on a probability space $(\Omega, \mathcal{F}, P)$.

   (a) Prove that

$$P\{\omega \in \Omega : \{X_1(\omega), X_2(\omega), \ldots\} \text{ is dense in } (0,1)\} = 1.$$

   (b) Using part (a) or otherwise, prove that

$$P\left(\bigcap_{(n_1, n_2, \ldots)} \{\omega \in \Omega : \lim_{k\to\infty} \frac{X_{n_1}(\omega) + \cdots + X_{n_k}(\omega)}{k} = \frac{1}{2}\} \right) = 0$$

   where the intersection is over all increasing sequences $(n_1, n_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$.

   (c) Did you just disprove the strong law of large numbers?

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Show that for any sequence of events $A_n \in \mathcal{F}$

   (a) $P\{A_n \text{ i.o.}\} \geq \limsup P(A_n)$.

   (b) $P\{A_n \text{ eventually}\} \leq \liminf P(A_n)$.

   (c) Give an example to show that one of these (which one?) fails if P is only assumed to be σ-finite.

4. Show that $X_n \to X$ a.s. if and only if $P(|X_n - X| > \epsilon \text{ i.o.}) = 0 \ \forall \epsilon > 0$.

5. Exercises 2.3.11 and 2.3.14 in Durrett (*Probability: Theory and Examples*, 4th edition).