1. (a) Let $X_1, X_2, \ldots$ be independent Unif$(-1,1)$ and $S_n = X_1^2 + \ldots + X_n^2$. Prove that

$$\frac{S_n - (n/3)}{\sqrt{n}} \quad \xrightarrow{d} \quad N(0, \sigma^2)$$

and find σ^2.

(b) Prove that

$$\sqrt{S_n} - \sqrt{\frac{n}{3}} \quad \xrightarrow{d} \quad N(0, \sigma_1^2)$$

and find σ_1^2.

(c) Let

$$A_n = \left\{ x \in \mathbb{R}^n \, : \, \sqrt{\frac{n}{3}} - 1 < \sqrt{x_1^2 + \ldots + x_n^2} < \sqrt{\frac{n}{3}} + 1 \right\}.$$

Show that for all sufficiently large n the Lebesgue measure of $A_n \cap (-1,1)^n$ is at least 99% of the Lebesgue measure of the cube $(-1,1)^n$.

2. (a) Let F be the number of fixed points of a uniform random permuation of $\{1, \ldots, n\}$. By counting the number of pairs (π, S) where π is a permutation of $\{1, \ldots, n\}$ and $S \subset \{1, \ldots, n\}$ is a set of k fixed points of π, find a formula for

$$E[F(F-1)...(F-k+1)].$$

(b) Using part a or otherwise, show that $E[F^k] = E[N^k]$ for all $k = 1, \ldots, n$, where $N \sim$ Pois(1).

3. Prove that if $N_i \sim$ Pois(λ_i) are independent with $\sum_{i=1}^{\infty} \lambda_i < \infty$, then $\sum_{i=1}^{\infty} N_i \sim$ Pois$(\sum_{i \geq 0} \lambda_i)$.

4. Exercises 3.3.3, 3.3.4, 3.3.12, 3.3.20, 3.4.5, 3.4.11, 3.6.12 in Durrett.