1. (Random Series) Fix $\beta \in \mathbb{R}$, and let
$$S_n = \sum_{k=1}^{n} k^\beta X_k.$$
where $X_1, X_2, \ldots$ are independent with $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$. For which values of β does S_n converge almost surely? Prove your answer.

2. (Arcsine Laws) Now take $\beta = 0$ so that $S_n = X_1 + \ldots + X_n$ is a simple random walk in $\mathbb{Z}$. Let
$$T_m = \inf\{n : \#\{1 \le k \le n : S_k = 0\} = m\}$$
be the time of the mth return to 0. Find the distribution of T_1.

3. (Stable Laws) Continuing the last problem, prove that there is an $\alpha > 0$ such that
$$\frac{T_m}{m^{1/\alpha}} \quad \stackrel{d}{\to} \quad Y$$
as $m \to \infty$, where Y is an α-stable random variable. What is α?

4. (Large Deviations) Continuing with $\beta = 0$, compare the two probabilities
$$P(S_n > \sqrt{n})$$
and
$$P(S_n > 0.01n).$$
Show that one of them (which one?) converges to a positive number as $n \to \infty$ and the other converges to 0. For the one that converges to 0, how fast is the convergence?

5. (Moment Problem) Prove that if a random variable Z satisfies
$$E[Z^{2n}] = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$$
for all $n \ge 1$ and $E[Z^{2n-1}] = 0$ for all $n \ge 1$, then Z has the $N(0,1)$ distribution.

Due Thursday 12/4