Let $G = (V, E)$ be a connected graph (assumed to be finite, except in problem 6) with positive edge conductances $(c(e))_{e \in E}$. Let $P = (p(x, y))_{x, y \in V}$ be the Markov transition matrix $p(x, y) = c(x, y)/C_x$, where $C_x = \sum_z c(x, z)$, and we define $c(x, y) = 0$ if $(x, y) \notin E$. Let $\pi(x) = C_x/C$ where $C = \sum_y C_y$.

1. In class we checked that $\pi$ is a left eigenvector of $P$ with eigenvalue 1. Use the maximum principle to show that $P$ has a unique right eigenvector with eigenvalue 1. Deduce that the stationary distribution is unique: $\pi$ is the only probability vector satisfying $\pi(x) = \sum_y \pi(y)p(y, x)$ for all $x \in V$.

2. Show that if $\iota$ is the unit current flow from $a$ to $z$, then $-\iota$ is the unit current flow from $z$ to $a$. Conclude that $R_{\text{eff}}(a \leftrightarrow z; G) = R_{\text{eff}}(z \leftrightarrow a)$.

3. Fix an edge $e$ and let $G/e$ be the graph obtained by gluing together the endpoints of $e$. Show that $R_{\text{eff}}(a \leftrightarrow z; G) \downarrow R_{\text{eff}}(a \leftrightarrow z; G/e)$ as $c(e) \uparrow \infty$.

In problems 4-6, assume $c(e) = 1$ for all $e \in E$.

4. Calculate $R_{\text{eff}}(1 \leftrightarrow 2)$ for the complete graph consisting of all $\binom{n}{2}$ edges on $V = \{1, 2, \ldots, n\}$.

5. In class we saw that distinct edges $e, f \in E$ are nonpositively correlated in the uniform spanning tree: $\Pr(e \in U, f \in U) \leq \Pr(e \in U)\Pr(f \in U)$. This problem is about when equality holds.

Write $e = (a, z)$ and $f = (x, y)$. Let $\iota^e$ be the unit current flow from $a$ to $z$. For $u \in V$ let $P_u$ be the law of simple random walk on $G$ started from $u$, and let $T_u$ be the first hitting time of $u$.

Prove that the following are equivalent:

(i) $\Pr(e \in U, f \in U) = \Pr(e \in U)\Pr(f \in U)$.
(ii) $\iota^e(f) = 0$.
(iii) $\Pr_x(T_a < T_z) = \Pr_y(T_a < T_z)$.

6. In this problem, $V$ is infinite and $G$ is locally finite. Let $(V_n)_{n \geq 1}$ be an exhaustion of $V$ and let $\tau_n = \inf\{k \geq 0 : X_k \notin V_n\}$. (By the way, what is our convention for infimum of the empty set? Explain why this convention makes sense!)

(a) Prove that $\Pr_a(\tau_n < \infty) = 1$ for all $n \geq 1$ and all $a \in V$.
(b) Prove that $\tau_n \uparrow \infty$. (That is, $\tau_n \leq \tau_{n+1}$ for all $n$, and $\lim \tau_n = \infty$.)

Hand in any four of the above six problems, plus any five of the following exercises from Lyons and Peres: 2.1a, 2.4, 2.12, 2.13 (Hint: what property does the minimizer $F$ have on the complement of $A \cup Z$?), 2.23, 2.43 (Hint: use the martingale convergence theorem!), 2.61, 2.66, 2.73, 4.6.

Due Friday Feb 16