Let $G = (V, E)$ be a connected graph (assumed to be finite, except in problem 6) with positive edge conductances $(c(e))_{e \in E}$. Let $P = (p(x, y))_{x,y \in V}$ be the Markov transition matrix $p(x, y) = c(x, y)/C_x$, where $C_x = \sum_z c(x, z)$, and we define $c(x, y) = 0$ if $(x, y) \notin E$. Let $\pi(x) = C_x/C$ where $C = \sum_y C_y$.

1. In class we checked that π is a left eigenvector of P with eigenvalue 1. Use the maximum principle to show that P has a unique *right* eigenvector with eigenvalue 1. Deduce that the stationary distribution is unique: π is the only probability vector satisfying $\pi(x) = \sum_{y \in V} \pi(y)p(y, x)$ for all $x \in V$.

2. Show that if ι is the unit current flow from a to z, then $-\iota$ is the unit current flow from z to a. Conclude that $R_{\text{eff}}(a \leftrightarrow z) = R_{\text{eff}}(z \leftrightarrow a)$.

3. Fix an edge e and let G/e be the graph obtained by gluing together the endpoints of e. Show that

$$R_{\text{eff}}(a \leftrightarrow z; G) \downarrow R_{\text{eff}}(a \leftrightarrow z; G/e) \qquad \text{as } c(e) \uparrow \infty.$$

In problems 4-6, assume $c(e) = 1$ for all $e \in E$.

4. Calculate $R_{\text{eff}}(1 \leftrightarrow 2)$ for the complete graph consisting of all $\binom{n}{2}$ edges on $V = \{1, 2, \ldots, n\}$.

5. In class we saw that distinct edges $e, f \in E$ are nonpositively correlated in the uniform spanning tree: $\mathbb{P}(e \in U,\ f \in U) \leq \mathbb{P}(e \in U)\mathbb{P}(f \in U)$. This problem is about when equality holds.

   Write $e = (a, z)$ and $f = (x, y)$. Let ι^e be the unit current flow from a to z. For $u \in V$ let $\mathbb{P}_u$ be the law of simple random walk on G started from u, and let T_u be the first hitting time of u.

   Prove that the following are equivalent:

   (i) $\mathbb{P}(e \in U,\ f \in U) = \mathbb{P}(e \in U)\mathbb{P}(f \in U)$.

   (ii) $\iota^e(f) = 0$.

   (iii) $\mathbb{P}_x(T_a < T_z) = \mathbb{P}_y(T_a < T_z)$.

6. In this problem, V is infinite and G is locally finite. Let $(V_n)_{n \geq 1}$ be an exhaustion of V and let $\tau_n = \inf\{k \geq 0 : X_k \notin V_n\}$. (By the way, what is our convention for infimum of the empty set? Explain why this convention makes sense!)

   (a) Prove that $\mathbb{P}_a(\tau_n < \infty) = 1$ for all $n \geq 1$ and all $a \in V$.

   (b) Prove that $\tau_n \uparrow \infty$. (That is, $\tau_n \leq \tau_{n+1}$ for all n, and $\lim \tau_n = \infty$.)

Hand in **any four** of the above six problems, plus **any five** of the following exercises from Lyons and Peres: 2.1a, 2.4, 2.12, 2.13 (*Hint: what property does the minimizer F have on the complement of $A \cup Z$?*), 2.23, 2.43 (*Hint: use the martingale convergence theorem!*), 2.61, 2.66, 2.73, 4.6.

Due Friday Feb 16