Let $G = (V, E)$ be a finite connected graph. The Ising, Potts, and Random Cluster measures are defined respectively on spin configurations $\eta \in \{-1, 1\}^V$, color configurations $\sigma \in \{1, \dots, q\}^V$, and percolation configurations $\omega \in \{0, 1\}^E$, by the formulas

$$\frac{1}{Z_I} e^{-\alpha H_I(\eta)}, \quad \frac{1}{Z_P} e^{-\beta H_P(\sigma)}, \quad \frac{1}{Z_{RC}} \prod_{(x,y)\in E} p^{\omega(x,y)}(1-p)^{1-\omega(x,y)} q^{k(\omega)}$$

where $H_I(\eta) = -\sum_{(x,y)\in E} \eta_x \eta_y$ and $H_P(\sigma) = -\sum_{(x,y)\in E} 1\{\sigma_x = \sigma_y\}$, and $k(\omega)$ is the number of clusters (connected components of the graph $(V, \omega^{-1}(1))$).

1. In class we showed that as $p, q \downarrow 0$ with $p = q$, the random cluster measure converges to the uniform spanning forest of G. This problem is about other limits (some more interesting than others!) Remember that the normalizing constant Z_{RC} depends on p and q; and Z_P depends on β and q.
   (a) What is the limit of the random cluster measure as $p, q \downarrow 0$ with $q/p \to 0$?
   (b) What is the limit of the random cluster measure as $p, q \downarrow 0$ with $p/q \to 0$?
   (c) What is the limit of the Potts measure as $\beta \to -\infty$?
   (d) What is the limit of the Potts measures as $\beta \to +\infty$?
2. Let $q = 2$ and $p = 1 - e^{-\beta}$. This problem comes from Hugo Duminil-Copin's lecture notes on the Ising and Potts models.
   (a) Find a change of variables that transforms the Ising measure into the $q = 2$ Potts measure.
   (b) Using the coupling of Potts and random cluster measures, show that for any $x, y \in V$,

   $$\mathbb{E}(\eta_x \eta_y) = \mathbb{P}\{x \text{ and } y \text{ belong to the same cluster of } \omega\}.$$

   Here and in the rest of this problem, expectations $\mathbb{E}$ refer to the Ising measure and probabilities $\mathbb{P}$ refer to the random cluster measure.
   (c) Show that for any subset A of V,

   $$\mathbb{E}\left(\prod_{x\in A} \eta_x\right) = \mathbb{P}\{\text{every cluster of } \omega \text{ intersects } A \text{ in a set of even cardinality}\}.$$

   (d) Show that for any subsets A and B of V,

   $$\mathbb{E}\left(\prod_{x\in A\cup B} \eta_x\right) \geq \mathbb{E}\left(\prod_{x\in A} \eta_x\right) \mathbb{E}\left(\prod_{x\in B} \eta_x\right).$$

   This is known as the *second Griffiths inequality*.

3. Comparison of p-norms, $1 \le p < \infty$.

   (a) Let π be a probability measure on a finite set V. For $f : V \to \mathbb{R}$, define

   $$\|f\|_{p,\pi} = \left(\sum_{x \in V} |f(x)|^p \pi(x) \right)^{1/p}.$$

   Show that $\|f\|_{p,\pi}$ is nondecreasing in p.

   (b) (ℓ^p space) Show that for sequences $x = (x_1, x_2, \ldots)$ of real numbers, the inequality goes the opposite way: $\|x\|_p := \left(\sum_{n\ge1} |x_n|^p\right)^{1/p}$ is nonincreasing in p.

   (c) ($L^p([0,1])$ Which way does the inequality go for functions on $[0,1]$, with $\|f\|_p := \left(\int |f|^p\right)^{1/p}$?

   (d) ($L^p([0,\infty))$ Show that neither inequality holds if we replace $[0,1]$ by $[0,\infty)$.

4. Let $(M_n)_{n\ge0}$ be a martingale such that $\mathbb{E}M_n^2 < \infty$ for all n.

   (a) Prove that the increments of M are uncorrelated:

   $$\mathbb{E}[(M_{k+1} - M_k)(M_{n+1} - M_n)] = 0 \qquad \text{for all } 0 \le k < n.$$

   (b) Let $A_n = \sum_{k=0}^{n-1} E[(M_{k+1} - M_k)^2 | \mathcal{F}_k]$ be the quadratic variation of M. Prove a strong law of large numbers,

   $$\frac{M_n(\omega)}{A_n(\omega)} \to 0 \qquad \text{for a.e. } \omega \text{ such that } A_n(\omega) \to \infty.$$

Hand in **any three** of problems 1–4 above, plus **any three** of the following exercises from Levin and Peres: 2.5, 2.6, 4.3, 4.4, 12.1ab, 12.2, 12.7 (the definition of $e(A,B)$ is missing $|\cdot|$ for cardinality), plus **any three** of the following exercises from Lyons and Peres: 5.20, 5.23, 5.25, 5.28, 5.33, 5.62, 5.64.