1. Let $X \in \mathbb{R}^d$ ($d \geq 2$) be a random vector with pairwise independent coordinates and a rotationally invariant distribution (i.e., UX has the same distribution as X for all orthogonal $d \times d$ matrices U). Prove that X has the multivariate normal distribution $N(0, \sigma^2 I_d)$ for some $\sigma > 0$.

2. Let X and Y be real-valued random variables with $\mathbb{E}X = \mathbb{E}Y = \mathbb{E}(XY) = 0$.

   (a) Prove or disprove: If both X and Y have normal distributions, then X and Y are independent.

   (b) Prove or disprove: If (X, Y) has a multivariate normal distribution, then X and Y are independent.

3. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion.

   (a) Show that $B_1 + B_2$ is normally distributed and find its mean and variance.
   (*Hint: Express it as a sum of independent normal random variables. The variance is not* 3.)

   (b) Find $P(B_1 < B_2 < B_3)$ and $P(B_1 < B_3 < B_2)$.

   (c) Let $0 < s < t < u < v$. Find formulas for $\mathbb{E}(B_s B_t)$ and $\mathbb{E}(B_s B_t B_u)$ and $\mathbb{E}(B_s B_t B_u B_v)$.
   (*Hint: To reduce the amount of calculation, look up Wick's formula!*)

   (d) Calculate the mean and variance of the maximum $M_t := \max_{0 \leq s \leq t} B_t$.
   (*You'll find that* $\mathbb{E}M_t = ct^a$ *and* $\operatorname{Var} M_t = c't^b$ *for constants* a, b, c, c'*. Test your intuition first by predicting the values of the exponents* a *and* b*. Then do the explicit calculation using* $M_t \stackrel{d}{=} |B_t|$.)

4. (**A local martingale**) Let $T = \inf\{t > 0 : B_t = 1\}$.

   (a) Prove that $\mathbb{P}(T < \infty) = 1$.

   (b) Let $X(t) = B_{\frac{t}{1-t} \wedge T}$ for $t \in [0, 1)$ and $X(t) = 1$ for $t \geq 1$. Prove that $\mathbb{E}X(t) = 0$ for all $t \in [0, 1)$.

   (c) Is $(X(t))_{t \geq 0}$ a martingale?

   (d) Prove that X is a **local martingale**: There is a sequence of stopping times $\tau_n \uparrow \infty$ such that for each n, the stopped process $(X(t \wedge \tau_n))_{t \geq 0}$ is a martingale.

   (e) Is it possible for a stochastic process X with continuous sample paths (i.e., the map $t \mapsto X(t)$ is a.s. continuous) to have discontinuous expectation (i.e., the map $t \mapsto \mathbb{E}X(t)$ is not continuous)?

5. Do Morters and Peres exercise 1.5. Your friend tells you that

   "*The Brownian bridge* X *from* 0 *to* y *has the same distribution as*
   a standard Brownian motion B *conditioned on* $B(1) = y$"

   and then leaves on a backpacking trip before you can ask her to clarify. Explain what is problematic about conditioning on $B(1) = y$. Then state and prove a precise version of her claim. (There is more than one way to make it precise!)

Hand in **any three** of problems 1–5 above, plus **any three** of the following exercises from Morters and Peres: 1.11, 1.12, 2.10, 2.12, 2.15, 2.19, plus **any two** of the following exercises from Durrett (Probability: Theory and Examples, 4th edition): 7.1.1, 7.1.2, 7.1.3.