HOW TO MAKE THE MOST OF A SHARED MEAL: PLAN THE LAST BITE FIRST.

LIONEL LEVINE AND KATHERINE E. STANGE

ABSTRACT. If you are sharing a meal with a companion, how best to make sure you get your favourite fork-fulls? Ethiopian Dinner is a game in which two players take turns eating morsels from a common plate. Each morsel comes with a pair of utility values measuring its tastiness to the two players. Kohler and Chandrasekaharan discovered a good strategy — a subgame perfect equilibrium, to be exact — for this game. We give a new visual proof of their result. The players arrive at the equilibrium by figuring out their last move first and working backward. We conclude that it's never too early to start thinking about dessert.

INTRODUCTION

Consider two friendly but famished acquaintances sitting down to dinner at an Ethiopian restaurant. The food arrives on a common platter, and each friend has his own favourite and not-so-favourite dishes among the spread. Hunger is a cruel master, and each of our otherwise considerate companions finds himself racing to swallow his favourites before his comrade can scoop them up. Each is determined to maximize his own gastronomic pleasures, and could not care less about the consequences for his companion.

An *Ethiopian Dinner* is a finite set

$$D = \{m_1, \ldots, m_n\}, \qquad m_i = (a_i, b_i)$$

whose elements are called *morsels*. Each morsel m_i is an ordered pair of real numbers (a_i, b_i). Two players, Alice and Bob, take turns removing one morsel from D and eating it. Each morsel can be eaten exactly once, and the game ends when all morsels have been eaten. Alice's score is the sum of the a_i for the morsels m_i she eats, while Bob's score is the sum of the b_i for

the morsels m_i he eats. We assume that the players' preferences are totally ordered, that is, $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$.

In such a game, the players are not adversaries; in fact, the game may end quite peaceably and successfully for both players if they have dissimilar tastes. The question we are interested in is this: if a player acts rationally to maximize her own score, and assumes that her meal partner does the same, what should be her strategy?

Eating your favourite morsel on the first move of an Ethiopian Dinner is not necessarily a good strategy. For example, if the dinner is

$$D = \{(1,2), (2,3), (3,1)\},$$

then Alice's favourite morsel is $(3,1)$. If she takes this morsel first, then Bob will take $(2,3)$, leaving Alice with $(1,2)$ for a total score of 4. Instead Alice should snag $(2,3)$ on the first move; after Bob takes $(1,2)$, Alice can finish up with $(3,1)$ for dessert and a total score of 5.

If deciding on the first move in an Ethiopian Dinner appears complicated, the *last move* is a different matter. The subject of this paper is a strategy discovered by Kohler and Chandrasekaharan [6], which we call the *crossout strategy*. Its mantra is:

> "*Eat your opponent's least favourite morsel on your own last move.*"

To arrive at this strategy, each player reasons informally as follows: My opponent will never choose her least favourite morsel, unless it is the only one left; therefore, unless this is my last move, I can safely save my opponent's least favourite morsel for later.

This reasoning predicts that if, say, Bob has the last move of the game, then Bob's last move will be to eat Alice's least favourite morsel. Because this is a game of perfect information, both players can use this reasoning to predict with certainty the game's last move. We now cross out Alice's least favourite morsel from the dinner D to arrive at a smaller dinner D' in which Alice has the last move. The same reasoning now implies that on her last move, Alice will eat Bob's least favourite morsel in D'. We then cross out Bob's least favourite morsel from D' and proceed inductively, alternately crossing out Alice's least favourite and Bob's least favourite among the remaining morsels until all morsels have been crossed out. The *crossout strategy* is to eat the *last* morsel to be crossed out.

What makes a strategy good? To convert the informal reasoning above into a proof that crossout is a "good" strategy, we need to define what makes a strategy good! The appropriate notion of good strategy depends on the class of games one is considering. Ethiopian Dinner is a nonzero-sum game:
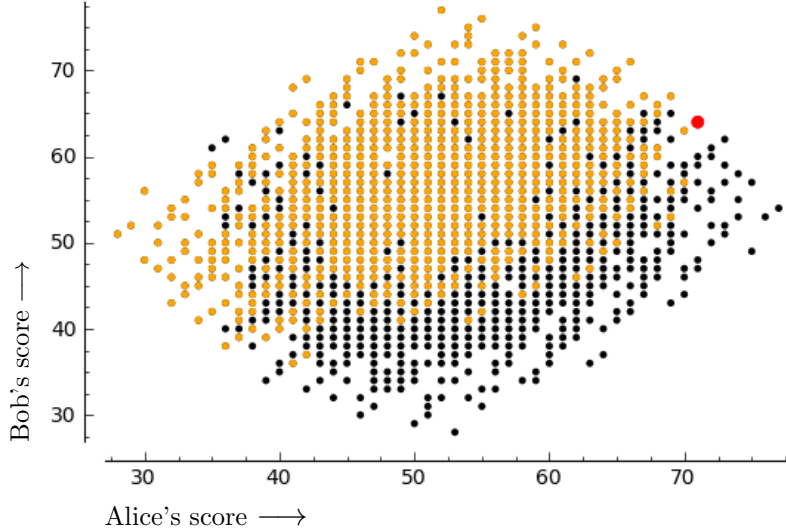
FIGURE 1. Plot of the score pairs for all possible outcomes of a permutation dinner D of size 14. The large red dot ● at upper right represents the score Alice and Bob receive if they both play the crossout strategy. The orange dots • represent scores for strategy pairs of the form (s, c) for s arbitrary: these are all of the outcomes Alice can obtain playing against Bob's crossout strategy. According to Theorem 1, among these outcomes she does best when she herself plays crossout. Black dots represent the outcomes of all other strategy pairs. Produced using Sage Mathematics Software [10].

one player's gain may not be the other's loss. In such a game, the basic requirement of any pair of good strategies (one for Alice, one for Bob) is that they form a *Nash equilibrium*, which means that neither player can benefit himself by changing strategies unilaterally.

A Nash equilibrium represents a stable, predictable outcome: Alice can declare, "I am playing my equilibrium strategy, and you'd do best to play yours." If Bob responds rationally by playing his own equilibrium strategy, then both players know how the game will turn out.

A game may have many equilibrium strategy pairs, some with better outcomes than others, so one tends to look for equilibria with further desirable properties. Which properties again depends on the class of games being considered. In the lingo of game theory, Ethiopian Dinner is a *perfect-information non-cooperative game in extensive form*. That is, both players

know the values $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ (perfect information); the players may not bargain or make side deals (non-cooperative); and the players alternate making moves (extensive form).

Non-cooperative games model situations in which the players have no way of communicating (perhaps our dinner guests don't speak a common language?) or are forbidden to collude. For instance, airlines are forbidden by law from colluding to fix prices. Colluding to fix the outcome of a meal is still legal in most countries, but Alice might nevertheless be dissuaded by cultural taboo from making propositions like "If you pay me fifty cents I promise not to eat any more spinach."

An Ethiopian Dinner with n morsels is certain to end in n moves. A widely accepted notion of a good strategy for games of this type (perfect information, non-cooperative, extensive form, finite length) is the *subgame perfect equilibrium*. This is a refinement of the Nash equilibrium which requires that the strategies remain in equilibrium when restricted to any subgame. In our case, a subgame is just a *subdinner* consisting of a subset of the morsels, with the same player moving last. A subgame perfect equilibrium is robust in the sense that even if one player, say Bob, makes a "mistake" on a particular move by deviating from his equilibrium strategy, Alice can confidently continue playing her equilibrium strategy because the same strategy pair is still an equilibrium of the resulting subgame. See, e.g., [8] and [9] for background on these concepts.

Let c be the crossout strategy described above for Ethiopian Dinner. We will give a new proof of the following theorem, which is due to Kohler and Chandrasekaharan [6].

Theorem 1. *The pair (c, c) is a subgame perfect equilibrium.*

In other words, if Alice plays crossout, then Bob cannot benefit himself by playing a different strategy; and vice versa.

Figure 1 illustrates Theorem 1 in the case of a particular *permutation dinner*, that is, a dinner of the form

$$D = \{(1, b_1), (2, b_2), \ldots, (n, b_n)\}$$

where $b_1, \ldots, b_n$ is a permutation of the numbers $1, \ldots, n$. Each dot in the figure represents the outcome of a strategy pair, with Alice's score plotted on the horizontal axis and Bob's score on the vertical axis, for the following permutation dinner of size 14:

$$D = \{(1, 6), (2, 14), (3, 10), (4, 3), (5, 7), (6, 5), (7, 9), (8, 8),$$
$$(9, 4), (10, 13), (11, 12), (12, 11), (13, 2), (14, 1)\}.$$

To visualize Theorem 1, note that the large red dot ● in Figure 1, which represents the outcome when both players play crossout, is rightmost among all possible outcomes achievable by Alice given that Bob plays crossout (such outcomes are indicated by orange dots ●).

Crossout is an efficiently computable equilibrium. In games arising in the real world, for instance in evolutionary dynamics and in economics, the appeal of the Nash equilibrium concept is twofold. First, it can explain why we observe certain strategies and not others. Second, even in the case of a game that has multiple equilibria and lacks a well-defined "best" outcome, knowing an explicit equilibrium provides certainty. Alice simply announces her intention to play crossout, refers Bob to the proof of Theorem 1 and trusts that his own best interest compels him to follow suit. What might have been a tense evening with an unpredictable outcome becomes a more relaxed affair in which each player can predict in advance which morsels she will be gobbling up.

To reap these benefits, the players must be able to compute an equilibrium pair, not just know that one exists! A recent strand of research, popularized by the slogan "*if your laptop can't find it, then, probably, neither can the market*," has explored the tendency for equilibria to be extremely difficult to compute [4]. The general existence proof for subgame perfect equilibria [8, VIII.2.10] uses a backward induction from the last move: if converted naively into an algorithm, it would seem to require searching through all possible move sequences in order to find an equilibrium. This kind of brute force search is typically out of the question even for games of moderate size. (For example, an Ethiopian Dinner of n morsels has $n!$ possible move sequences.) For this reason, it is always interesting to identify special classes of games that have efficiently computable equilibria. The crossout equilibrium for Ethiopian Dinner is an example: if both players play the crossout strategy, then they eat the morsels in reverse order of the crossouts. In this case, the entire move sequence of the dinner can be worked out in the order $n \log n$ time it takes to sort the two lists $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$.

PROOF OF EQUILIBRIUM

Dinners and strategies. A *dinner* is a finite set of morsels

$$D = \{m_1, \ldots, m_n\}.$$

Each morsel $m \in D$ comes with a pair of real numbers $u_A(m), u_B(m)$ representing its utility to Alice and Bob. We often write m as an ordered pair,

$$m = (u_A(m), u_B(m)).$$

We adopt the convention that *Bob has the final move* by default. Since moves alternate, the first move is determined by the parity of n: Alice has the first move if n is even, and Bob has the first move if n is odd.

A *strategy* is a map assigning to any non-empty dinner D a morsel $s(D) \in D$ to be eaten by the first player. Suppose that $P \in \{\text{Alice, Bob}\}$ is a player, D is a dinner, and that P plays strategy s. If it is P's turn to move, he selects morsel $s(D)$ and receives payoff $u_P(s(D))$. The remaining dinner is $D - s(D)$, with his opponent to move. Suppose his opponent plays strategy t. Play continues in this manner, and the *score* $v_P^D(s,t)$ of player P is defined by the recurrence

$$v_P^D(s,t) = \begin{cases} v_P^{D-s(D)}(s,t) + u_P(s(D)) & \text{if } P \text{ plays first in } D, \\ v_P^{D-t(D)}(s,t) & \text{if } P \text{ plays second in } D, \\ 0 & \text{if } D = \emptyset. \end{cases} \qquad (1)$$

where for $m \in D$, the dinner $D - m$ denotes D with morsel m removed. Since D has finitely many morsels, equation (1) defines $v_P^D(s,t)$ uniquely.

Our convention in denoting a player's score is that his own strategy is always the first listed in the ordered pair.

Formally, we can regard Ethiopian Dinner as a single game whose positions comprise all finite dinners. A pair of strategies (s,t) is a *subgame perfect equilibrium* for this game if

$$v_A^D(s',t) \le v_A^D(s,t) \quad \text{and} \quad v_B^D(t',s) \le v_B^D(t,s)$$

for all strategies s' and t' and all finite dinners D.

The Crossout Strategy. After giving the formal definition of the crossout strategy described in the introduction, we explain how to visualize it using a "crossout board" and prove the lemma that lies at the heart of our argument, the Crossout Board Lemma (Lemma 2).

Let D be a set of n morsels. Write $\ell_A(D)$ for Alice's least favourite morsel in D, and $\ell_B(D)$ for Bob's least favourite morsel in D. Let $D_1 = D$, and

$$D_{i+1} = D_i - m_i, \qquad i = 1, \ldots, n-1$$

where

$$m_i = \begin{cases} \ell_A(D_i), & i \text{ odd} \\ \ell_B(D_i), & i \text{ even}. \end{cases}$$

The sequence of morsels

$$m_1, m_2, \ldots, m_n$$

is called the *crossout sequence* of D. Note that

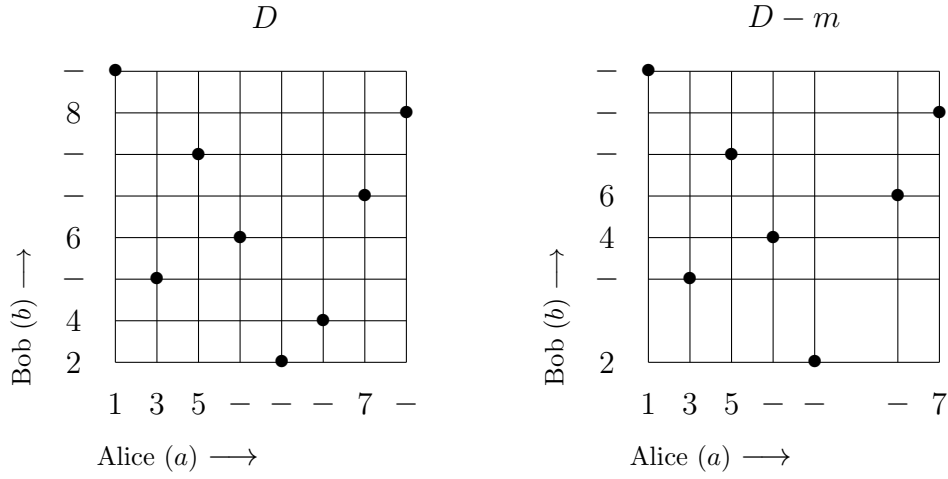$$m_1 \text{ is Alice's least favourite morsel in } D$$

FIGURE 2. Left: Example of a crossout board for a dinner D with 8 morsels. Labels on the axes indicate the crossout sequence. Right: The crossout board for the dinner $D - m$, in which a morsel m has been removed. By the Crossout Board Lemma, each label on the right is at least as far from the origin as the corresponding label on the left.

m_2 is Bob's least favourite morsel in $D - m_1$

m_3 is Alice's least favourite morsel in $D - m_1 - m_2$

m_4 is Bob's least favourite morsel in $D - m_1 - m_2 - m_3$

$\dots$

Now suppose D is a dinner, i.e., a set of n morsels with Bob distinguished to move last. The *crossout strategy* c is defined by $c(D) = m_n$. Note that if both players play the crossout strategy, then they eat the morsels in *reverse order* of the crossout sequence:

$$m_n = c(D)$$
$$m_{n-1} = c(D - m_n)$$
$$m_{n-2} = c(D - m_n - m_{n-1})$$
$$\vdots$$
$$m_1 = c(D - m_n - \dots - m_2).$$

Thus m_1, which is Alice's least favorite morsel in D, is eaten by Bob on the last turn.

Crossout boards. To prepare for the proof of Theorem 1, it is convenient to illustrate the crossout sequence with a *crossout board*, as in Figure 2. We display the dinner on a Cartesian coordinate plane: each morsel $m = (a, b)$ is graphed as a dot at coordinate (a, b). Since we assume that the players' preferences are totally ordered, each vertical or horizontal line passes through at most one morsel. The crossout sequence itself is indicated by writing the number (or *label*) i on the a-axis below m_i if i is odd, and on the b-axis to the left of m_i if i is even.

Figure 2 shows the crossout board of the dinner

$$D = \{(1,8), (2,3), (3,6), (4,4), (5,1), (6,2), (7,5), (8,7)\}$$

and of $D - m$, where m is the morsel $(6, 2)$. It is helpful to imagine placing the labels on a crossout board one at a time in increasing order. Alice starts at the left and scans rightward, placing the label 1 below her least favorite morsel. Then Bob starts at the bottom and scans upward, placing the label 2 to the left of his least favorite unlabeled morsel. The players alternate in this fashion until all morsels are labeled. Note that the labels on each axis appear in increasing order moving away from the origin. Alice always performs the first crossout, because of our convention that Bob has the last move. Hence, the odd labels appear on Alice's axis and the even labels on Bob's axis.

The central lemma needed to show that crossout is an equilibrium is the following.

Lemma 2 (Crossout Board Lemma). *Let D be a dinner, and $\widehat{D} \subset D$ a subdinner. For each $k = 1, \ldots, |\widehat{D}|$ the location of label k in the crossout board of $\widehat{D}$ is at least as far from the origin as the location of label k in the crossout board of D.*

Proof. Let B be the crossout board for D, with crossout sequence

$$m_1, m_2, \ldots, m_{|D|}.$$

Let $\widehat{B}$ be the crossout board for $\widehat{D}$, with crossout sequence

$$\widehat{m}_1, \widehat{m}_2, \ldots, \widehat{m}_{|\widehat{D}|}.$$

For morsels p and q of D, we write $p <_D q$ to mean that p appears before q in the crossout sequence for D. If p and q are also morsels of $\widehat{D}$, then we write $p <_{\widehat{D}} q$ to mean that p appears before q in the crossout sequence for $\widehat{D}$. In particular, for any $1 \le j, k \le |\widehat{D}|$ we have

$$m_j \underset{D}{<} m_k \iff j < k \iff \widehat{m}_j \underset{\widehat{D}}{<} \widehat{m}_k. \tag{2}$$
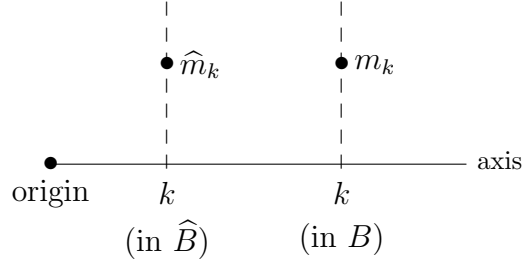
FIGURE 3. If label k is closer to the origin in $\widehat{B}$ than in B, then $\widehat{m}_k$ must have been labeled already in B by some $j < k$, i.e. $\widehat{m}_k <_D m_k$.

Given $1 \leq k \leq |\widehat{D}|$ let us say k is *jumpy* if the label k is strictly closer to the origin in $\widehat{B}$ than in B. We will show that there are no jumpy labels.

Let P be the player who places the label k (so P is Alice if k is odd, Bob if k is even). When P places the label k on board B next to the morsel m_k, this morsel is the closest available to the origin along P's axis. If k is jumpy, then the morsel $\widehat{m}_k$ is closer to the origin along P's axis, which means that $\widehat{m}_k$ is *unavailable*, that is, it was already labeled in B by some $j < k$ (Figure 3). Hence

$$k \text{ is jumpy} \implies \widehat{m}_k \underset{D}{<} m_k. \tag{3}$$

Note also that if $\widehat{m}_k <_{\widehat{D}} m_k$, then in the crossout sequence for $\widehat{B}$, both $\widehat{m}_k$ and m_k are available at step k and $\widehat{m}_k$ is chosen. Therefore, the label k is placed closer to the origin in $\widehat{B}$ than in B. Hence

$$\widehat{m}_k \underset{\widehat{D}}{<} m_k \implies k \text{ is jumpy}. \tag{4}$$

Now suppose for a contradiction that one of the labels $1, \ldots, |\widehat{D}|$ is jumpy, and let k be the smallest jumpy label. Since $\widehat{m}_k \in \widehat{D}$ and $\widehat{D} \subset D$, the morsel $\widehat{m}_k$ also belongs to D. Let j be its label on the crossout board of D; that is,

$$m_j = \widehat{m}_k. \tag{5}$$

Then

$$
\begin{aligned}
k \text{ is jumpy} &\implies \widehat{m}_k \underset{D}{<} m_k && \text{by (3)} \\
&\implies m_j \underset{D}{<} m_k && \text{by (5)} \\
&\implies j < k && \text{by (2)}
\end{aligned}
$$

$$\implies \widehat{m}_j \underset{\widehat{D}}{<} \widehat{m}_k \qquad\qquad \text{by (2)}$$

$$\implies \widehat{m}_j \underset{\widehat{D}}{<} m_j \qquad\qquad \text{by (5)}$$

$$\implies j \text{ is jumpy} \qquad\qquad \text{by (4)}$$

That is, $j < k$ and j is jumpy. But k was the smallest jumpy label. This contradiction shows that there are no jumpy labels, completing the proof. □

The *crossout scores* $\chi_A(D)$ and $\chi_B(D)$ are the scores for Alice and Bob when both play the crossout strategy; that is,

$$\chi_A(D) = v_A^D(c, c) = m_2 + m_4 + \cdots + m_{2\lfloor n/2 \rfloor}$$
$$\chi_B(D) = v_B^D(c, c) = m_1 + m_3 + \cdots + m_{2\lceil n/2 \rceil - 1}.$$

These scores are easy to read off from the crossout board: The *unlabeled* morsel locations on a player's axis are precisely the utilities of the morsels he eats if both players follow the crossout strategy. Therefore, the crossout scores $\chi_A(D)$ and $\chi_B(D)$ are obtained by summing the unlabeled locations (marked with dashes in Figure 2) on the a- and b-axes respectively. For instance, for the board D pictured in Figure 2, we have $\chi_A(D) = 4+5+6+8$ and $\chi_B(D) = 3 + 5 + 6 + 8$.

If we also wish to show the order of play, we can label the a-coordinate of the morsel eaten by Alice in turn i with the symbol A_i, and the b-coordinate of the morsel eaten by Bob in turn j with the symbol B_j as shown in Figure 4. Alice's score is the sum of the a-coordinates labeled with A's, and Bob's score is the sum of the b-coordinates labeled with B's.

In D of Figure 4 we see that Alice, who plays first, eats her favourite morsel $(8, 7)$ on her first turn. In the remaining game $D - (8, 7)$, Bob moves first but does not eat his favourite morsel $(1, 8)$ until his last move (for such is Alice's loathing for it that he can safely ignore it until the end). An interesting property of the crossout strategy, which we leave as an exercise to the reader since it is not needed for the proof of the main theorem, is that if both players follow it, then the first player eventually eats her favourite morsel.

The Main Lemma. The next lemma shows that neither player can improve his crossout score by choosing a different first morsel.

Lemma 3 (Main Lemma)**.** *Let D be a dinner, and let m be a morsel of D. Let P be the player to move first in D. Then*
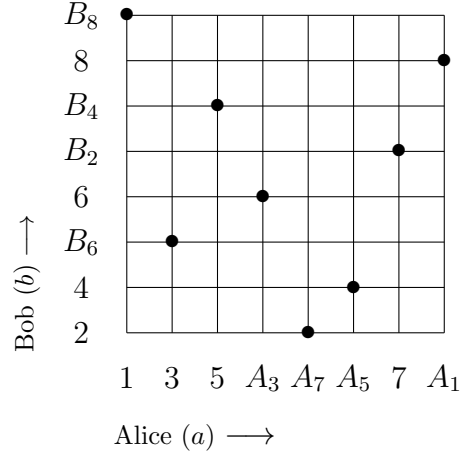
$$u_P(m) + \chi_P(D - m) \le \chi_P(D).$$

FIGURE 4. A crossout board showing the sequence of play: Alice eats the morsel above the label A_i on turn i, and Bob eats the morsel to the right of the label B_j on turn j.

Proof. We compare the crossout boards for D and $D - m$. (An example is illustrated in Figure 2.) In the latter, a morsel has been removed. Player P is the second player to move in $D - m$, so he swallows one fewer morsel in $D - m$ than in D. This means the number of labels on P's axis is the same in the crossout boards of D and $D - m$. By the Crossout Board Lemma 2, each label on the board for $D - m$ is no closer to the origin than the corresponding label on the board for D. Therefore the sum of the labeled positions on P's axis is at least as large in $D - m$ as in D. Hence the sum of the *unlabeled* positions on P's axis is no larger in $D - m$ than in D. For D, this sum is the crossout score $\chi_P(D)$. For the board $D - m$, this sum consists of the score $\chi_P(D - m)$ plus the utility $u_P(m)$ of the removed morsel m. $\square$

Proof of Theorem 1. Let D be a dinner of n morsels, and let c be the crossout strategy. We induct on n to show that for any player $P \in \{A, B\}$ and any strategy s,

$$v_P^D(s, c) \le v_P^D(c, c).$$

The base case $n = 1$ is trivial because c is the only strategy: In a game with one morsel, the only thing you can do is eat it!

On to the inductive step: Suppose first that P is the first player to move in D. Let $m = s(D)$. Then

$$v_P^D(s, c) = u_P(m) + v_P^{D-m}(s, c) \qquad \text{by (1)}$$
$$\le u_P(m) + v_P^{D-m}(c, c) \qquad \text{by the inductive hypothesis}$$

$$\leq v_P^D(c,c) \qquad\qquad\qquad\qquad \text{by Main Lemma 3.}$$

It remains to consider the case that P is the second player to move in D. Letting $m = c(D)$, we have by the inductive hypothesis and (1),

$$v_P^D(s,c) = v_P^{D-m}(s,c) \leq v_P^{D-m}(c,c) = v_P^D(c,c)$$

which completes the proof.

Concluding Remarks

We have analyzed Ethiopian Dinner as a non-cooperative game, and found an efficiently computable subgame perfect equilibrium, the crossout strategy. Here we discuss its efficiency, and mention some variants and generalizations.

Pareto Inefficiency. A strategy pair (s,t) is called *Pareto inefficient* if there exists another pair (s',t') that results in at least as good an outcome for both players and a strictly better outcome for one of them. An equilibrium may be Pareto inefficient, as demonstrated by the famous Prisoner's Dilemma, in which both players do better by mutual cooperation than by mutual defection even though mutual defection is the unique equilibrium [8].

For the permutation dinner shown in Figure 1, we see that (c,c) is Pareto efficient because there are no dots lying (weakly) both above and to the right of the crossout score ($\bullet$). However, (c,c) is not Pareto efficient in general. Among permutation dinners, the smallest counterexamples occur for dinners of size 6, for which there are two:

$$\{(1,5),(2,1),(3,2),(4,3),(5,4),(6,6)\},$$
$$\{(1,5),(2,1),(3,2),(4,4),(5,6),(6,3)\}.$$

To see how common this phenomenon is, we used Sage Mathematics Software [10] to check 10 000 randomly generated permutation dinners of size 16 for Pareto efficiency: All but 672 were Pareto efficient, and all but 241 were weakly Pareto efficient (that is, no other strategy pair resulted in strict improvements for both players). The improvement in scores achieved by alternate strategies was small: the largest improvement for any player was less than 8%. These findings provide some evidence that the crossout strategy is reasonably efficient. See [2, Theorem 1] for a proof that crossout *is* Pareto efficient with respect to a natural partial order on outcomes.

Generalized payoffs. The outcome of an Ethiopian Dinner is a partition of the index set $\{1,\ldots,n\}$ into a set $A = \{i_1,\ldots,i_{\lfloor n/2\rfloor}\}$ of $\lfloor n/2\rfloor$ morsels eaten by Alice and a set $B = \{j_1,\ldots,j_{\lceil n/2\rceil}\}$ of $\lceil n/2\rceil$ morsels eaten by Bob.

We have assumed that the final scores (payoffs) for Alice and Bob take the form

$$p_A = \sum_{i \in A} a_i, \qquad p_B = \sum_{j \in B} b_j.$$

This particular payoff function is not essential for the argument, however. Let

$$f_A : \mathbb{R}^{\lfloor n/2 \rfloor} \to \mathbb{R}, \qquad f_B : \mathbb{R}^{\lceil n/2 \rceil} \to \mathbb{R}$$

be functions that are strictly increasing in each coordinate, and symmetric with respect to permutations of the coordinates. Then the Ethiopian Dinner game with payoffs

$$p_A = f_A(a_{i_1}, \ldots, a_{i_{\lfloor n/2 \rfloor}}), \qquad p_B = f_B(b_{j_1}, \ldots, b_{j_{\lceil n/2 \rceil}})$$

has crossout as its optimal strategy. Indeed, the proof we have given uses only the relative order of the a_i and the b_j, and not their actual values.

One could also generalize the payoff function so that Alice's payoff depends not only on the morsels she ate but also on the morsels Bob ate, and vice versa. A natural choice is

$$p_A = \sum_{i \in A} a_i + \beta \sum_{j \in B} b_j,$$

$$p_B = \alpha \sum_{i \in A} a_i + \sum_{j \in B} b_j.$$

That is, Alice's payoff is the sum of her own utilities of the morsels she ate, plus β times the sum of the utilities to Bob of the morsels Bob ate. Bob's payoff is defined similarly. The parameters α and β measure the "friendliness" or degree of common interest of the players. The scenario of friends eating in an Ethiopian restaurant might correspond to values of α and β strictly between 0 and 1. One can also imagine scenarios with $\beta > 1$: perhaps Alice is Bob's mother and the morsels in question are brussels sprouts.

All of these games turn out to be equivalent to Ethiopian Dinner. Suppose we are considering the payoffs p_A and p_B on the dinner D consisting of morsels $m_i = (a_i, b_i)$ for $i = 1, \ldots, n$. Translating all of a player's utilities by an additive constant has no effect on strategy, so we may assume that $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 0$. Then

$$\sum_{j \in B} b_j = -\sum_{i \in A} b_i, \qquad \sum_{i \in A} a_i = -\sum_{j \in B} a_j.$$

Now let D' be the dinner consisting of morsels

$$m'_i = (a_i - \beta b_i, \, b_i - \alpha a_i), \qquad i = 1, \ldots, n.$$

Any strategy s on D has a corresponding strategy s' on D' (which chooses m_i' whenever s chooses m_i), and

$$p_P^D(s,t) = v_P^{D'}(s',t')$$

for both players $P \in \{A, B\}$. In other words, the modified payoff in D equals the usual Ethiopian Dinner payoff in D'. Therefore the pair (s, s) is an equilibrium for D, where s is such that $s' = c$ is the crossout strategy on D'.

We distinguish two extreme cases:

If $\alpha = \beta = -1$, then the game is zero-sum. In the terminology of combinatorial game theory, each morsel $m = (a, b)$ is a *switch* $\{a| - b\}$, so the full game is a sum of switches. The morsel (a, b) has temperature $a + b$, and optimal play proceeds in order of decreasing temperature (see [1] for background). The equivalent Ethiopian Dinner D' has morsels $(a_i + b_i, a_i + b_i)$ of equal appeal to both players, and crossout on D' gives the same decreasing-temperature play.

If $\alpha = \beta = 1$, then the game is fully cooperative. Both players have the same goal of maximizing their joint welfare. Since the game rules constrain them to alternate moves, the optimal play is the following: order the morsels $m_1, \ldots, m_n$ so that $a_i - b_i$ is a decreasing function of i. Alice takes morsels $m_1, \ldots, m_{\lfloor n/2 \rfloor}$, and Bob takes morsels $m_{\lfloor n/2 \rfloor+1}, \ldots, m_n$. In this case, the equivalent Ethiopian Dinner D' has morsels $(a_i - b_i, b_i - a_i)$ and crossout on D' gives the optimal strategy just described. Figure 5 shows examples of crossout boards for a zero-sum (competitive) dinner and a cooperative dinner.

We can measure the "cooperativeness" of a permutation dinner by its inversions. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a permutation of $1, \ldots, n$. For each pair of indices $i < j$ such that $\pi_i > \pi_j$, we call the pair (i, j) a *left inversion* of π and the pair (π_i, π_j) a *right inversion* of π. Both players should be pleased with a permutation dinner if it has a lot of inversions, because each inversion represents a pair of morsels m_i, m_j such that Alice prefers m_j while Bob prefers m_i. Hopkins and Jones [5] show that if the left inversions of π are a subset of the left inversions of π', then Alice's crossout score for the permutation dinner π' is at least as good as for π. In fact they show more: there is a bijection between the set of morsels Alice eats in π and the set she eats in π' such that each morsel eaten in π' is at least as tasty to Alice as corresponding one in π. (Alice prefers a prime piece of pie to an ordinary one: after all, who wouldn't?) Likewise, Bob prefers dinners with a lot of right inversions. (Curiously, although right inversions are in bijection with left inversions, set inclusion of right inversions induces a different partial
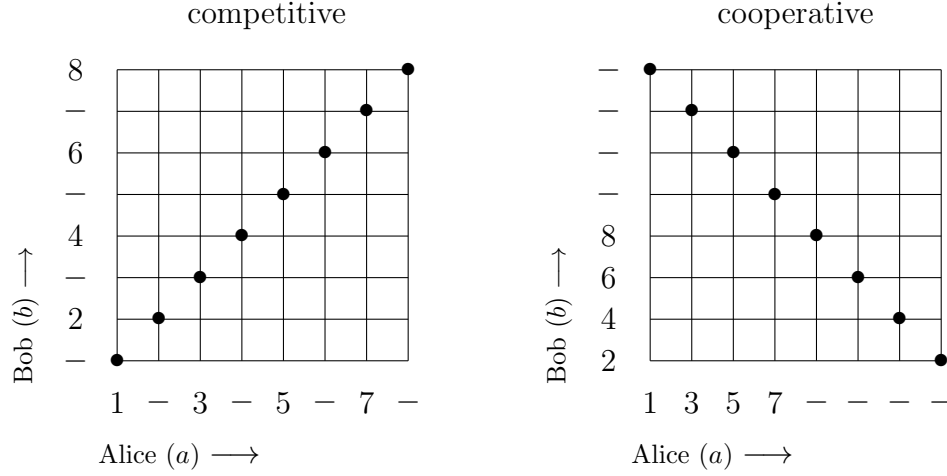
FIGURE 5. Crossout boards for a fully competitive and fully cooperative dinner.

ordering on permutations than does set inclusion of left inversions, as the reader can verify for permutations of 3 elements!)

Cake cutting and envy-free division. There is a large literature on *cake-cutting* [3] in which a cake (identified with the interval $[0, 1]$) comes equipped with a measure for each player describing the utility to him of eating a given piece. One problem is to find an *envy-free* partition of the cake, which means that each player prefers the piece assigned to him over the pieces assigned to the other players. When the cake is comprised of indivisible slices, this criterion becomes impossible to achieve in general, and finding an envy-minimizing allocation is a hard computational problem [7]. The outcome of the crossout strategy is reasonably close to envy-free: the first player is not envious, and the second player's envy is bounded by the utility of his favourite morsel. Another way of achieving an approximately envy-free allocation is described in [7, Theorem 2.1]. The algorithm described there is even faster than crossout, because it does not require sorting the lists of utilities.

OPEN QUESTIONS

We conclude by describing a few natural variants that we do not know how to analyze.

Delayed gratification. Suppose that the utilities $u_A(m)$ and $u_B(m)$ depend not only on the morsel m but on when it is eaten. A natural choice is to

value a morsel eaten on turn i with λ^i times its usual value, for a parameter $\lambda < 1$. Thus, a morsel declines in value the longer it remains on the plate (perhaps the delicate flavours are fading). The choice of exponential decay λ^i corresponds to the common assumption in economics that a payoff received in the future should be discounted to its net present value according to the prevailing interest rate: if the interest rate is α, then $\lambda = 1/(1 + \alpha)$. In the resulting game, each player feels an urgency to eat her favourites early on. Because time-sensitive payoffs break the symmetry assumption, our proof of equilibrium does not apply. Can the crossout strategy be modified to produce an equilibrium?

Inaccessible morsels. Ethiopian food is served atop *injera*, a layer of spongy bread that can only be eaten once it is revealed. If the game is played with the requirement that the order of consumption must respect a fixed partial ordering on the morsels, the crossout strategy may not be an allowable strategy. What should take its place?

Three's a crowd. Ethiopian Dinner resembles the process of draft picks in sports: each team participating in the draft has its own belief about how much each player is worth, and the teams draft players one at a time according to some predetermined order of play. Typically, many teams (more than two!) participate in the draft. Brams and Straffin [2] point out a number of pathologies in the case when the number of teams is greater than two: for example, it may be to a team's advantage to choose later in the draft. The crossout strategy for Ethiopian Dinner does not seem to apply when three or more people are sharing the meal, so we end with the following question: Is there an efficiently computable equilibrium for Ethiopian Dinner with three or more players?

We especially thank Asmara Restaurant in Cambridge, MA for the shared meal that inspired this paper. (The clear winner on that occasion was the first author's wife, whose strategy was to ignore all talk of game theory and eat all of her favourite morsels while her dinner companions were distracted by analyzing the game.)

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Department of Mathematics, MIT, 77 Massachusetts Ave., Cambridge, MA 02139. http://math.mit.edu/~levine
  E-mail address: levine@math.mit.edu

Department of Mathematics, Stanford University, 450 Serra Mall, Building 380, Stanford, CA 94305. http://math.katestange.net
  E-mail address: stange@math.stanford.edu