Chip-Firing and A Devil's Staircase

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Talk Outline

- Mode locking in dynamical systems.
- Discrete: parallel chip-firing.
- Continuous: iteration of a circle map $S^1 \to S^1$.
- How the devil's staircase arises.
- Short period attractors.

Mode Locking in Dynamical Systems

- "Weakly coupled oscillators tend to synchronize their motion, i.e. their modes of oscillation acquire $\mathbb{Z}$-linear dependencies."
  - J. C. Lagarias, 1991.
- Examples:
  - Huygens' clocks.
  - Solar system (rotational periods of moons and planets).
  - Biological oscillators: pacemaker cells, fireflies.
  - ...
- Parallel chip-firing: A combinatorial model of mode locking.

Parallel Chip-Firing on K_n

- At time t, each vertex $v \in [n]$ has $\sigma_t(v)$ chips
- If $\sigma_t(v) \geq n$, the vertex v is unstable, and fires by sending one chip to every other vertex.
- **Parallel update rule**: At each time step, all unstable vertices fire simultaneously:

$$\sigma_{t+1}(v) = \begin{cases} \sigma_t(v) + u_t, & \text{if } \sigma_t(v) \leq n-1 \\ \sigma_t(v) - n + u_t, & \text{if } \sigma_t(v) \geq n \end{cases}$$

where

$$u_t = \#\{v | \sigma_t(v) \geq n\}$$

is the number of unstable vertices at time t.

Parallel vs. Ordinary Chip-Firing

- In ordinary chip-firing (**Björner-Lovász-Shor**, **Biggs**, ...) one vertex is singled out as the sink. The sink is not allowed to fire.
- In parallel chip-firing, all vertices are allowed to fire.
  - $\Rightarrow$ The system may never reach a stable configuration.
- Instead of studying properties of the final configuration, we study properties of the dynamics.

The activity of a chip configuration

- Object of interest: The **activity** of σ is defined as

$$a(\sigma) = \lim_{t\to\infty} \frac{\alpha_t}{nt}$$

where

$$\alpha_t = u_0 + \ldots + u_{t-1}$$

is the total number of firings before time t.

- Since $0 \leq \alpha_t \leq nt$, we have $0 \leq a(\sigma) \leq 1$.

An Example on K_{10}

- $$\begin{array}{lrrrrrrrrrrl} \sigma_0 = (& 6 & 6 & 7 & 7 & 8 & 8 & 9 & 9 & 10 & 10) & \\ \sigma_1 = (& 8 & 8 & 9 & 9 & 10 & 10 & 11 & 11 & 2 & 2) & \\ \sigma_2 = (& 12 & 12 & 13 & 13 & 4 & 4 & 5 & 5 & 6 & 6) & \\ \sigma_3 = (& 6 & 6 & 7 & 7 & 8 & 8 & 9 & 9 & 10 & 10) & = \sigma_0 \end{array}$$
- Period 3, activity $1/3$.

- $$\begin{array}{lrrrrrrrrrrl} \sigma_0 = (& 7 & 7 & 8 & 8 & 9 & 9 & 10 & 10 & 11 & 11) & \\ \sigma_1 = (& 11 & 11 & 12 & 12 & 13 & 13 & 4 & 4 & 5 & 5) & \\ \sigma_2 = (& 7 & 7 & 8 & 8 & 9 & 9 & 10 & 10 & 11 & 11) & = \sigma_0 \end{array}$$
- Period 2, activity $1/2$.

How Does Adding More Chips Affect the Activity?

3	3	4	4	5	5	6	6	7	7	activity 0
4	4	5	5	6	6	7	7	8	8	activity 0
5	5	6	6	7	7	8	8	9	9	activity 0
6	6	7	7	8	8	9	9	10	10	activity $1/3$
7	7	8	8	9	9	10	10	11	11	activity $1/2$
8	8	9	9	10	10	11	11	12	12	activity $1/2$
9	9	10	10	11	11	12	12	13	13	activity $2/3$
10	10	11	11	12	12	13	13	14	14	activity 1
11	11	12	12	13	13	14	14	15	15	activity 1
12	12	13	13	14	14	15	15	16	16	activity 1

An Example on K_{100}

- Let $\sigma = (25\,25\,26\,26\,\ldots\,74\,74)$ on K_{100}.
- $(a(\sigma+k))_{k=0}^{100}$ =(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/6, 1/5, 1/5, 1/4, 1/4, 1/4, 2/7, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 2/5, 2/5, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 3/5, 3/5, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 5/7, 3/4, 3/4, 3/4, 4/5, 4/5, 5/6, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1).

An Example on K_{1000}

- Let $\sigma = (250\ 250\ 251\ 251\ \dots 749\ 749)$ on K_{1000}.
- $(a(\sigma + k))_{k=0}^{1000} =$

(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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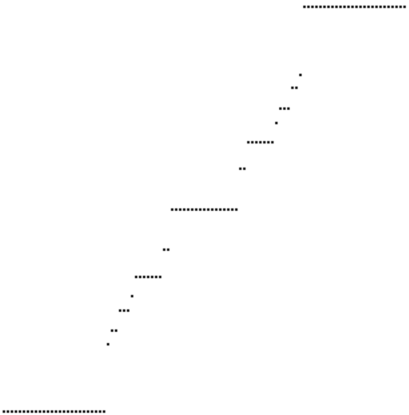
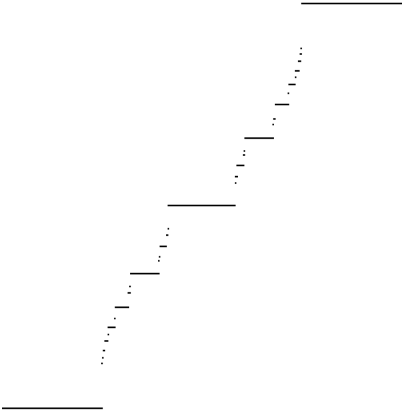
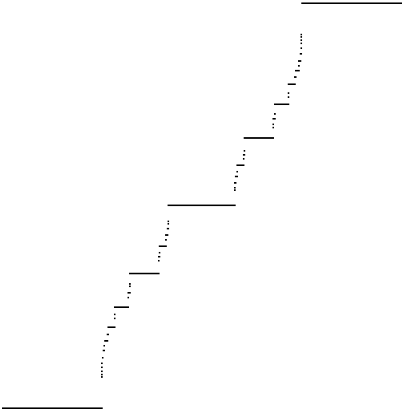
K_{10}



K_{100}



K_{1000}



K_{10000}

Questions

- Why such small denominators?
- Is there a limiting behavior as $n \to \infty$?

The Large n Limit

- Sequence of stable chip configurations $(\sigma_n)_{n \geq 2}$ with σ_n defined on K_n.
- **Activity phase diagram** $s_n : [0,1] \to [0,1]$

$$s_n(y) = a(\sigma_n + \lfloor ny \rfloor).$$

- Main hypothesis: $\exists$ continuous $F : [0,1] \to [0,1]$, such that for all $0 \leq x \leq 1$

$$\frac{1}{n} \# \{ v \in [n] \, | \, \sigma_n(v) < nx \} \to F(x)$$

as $n \to \infty$.

Main Result: The Devil's Staircase

- **Theorem** (LL, 2008): There is a continuous, nondecreasing function $s : [0,1] \to [0,1]$, depending on F, such that for each $y \in [0,1]$

$$s_n(y) \to s(y) \qquad \text{as } n \to \infty.$$

  Moreover
  - If $y \in [0,1]$ is irrational, then $s^{-1}(y)$ is a point.
  - For "most" choices of F, the fiber $s^{-1}(p/q)$ is an interval of positive length for each rational number $p/q \in [0,1]$.
- So for most F, the limiting function s is a *devil's staircase*: it is locally constant on an open dense subset of $[0,1]$.
- Stay tuned for:
  - The construction of s.
  - What "most" means.

From Chip-Firing to Circle Map

- Call σ confined if
  - $\sigma(v) \leq 2n-1$ for all vertices v of K_n;
  - $\max_v \sigma(v) - \min_v \sigma(v) \leq n-1$.
- **Lemma**: If $a(\sigma_0) < 1$, then there is a time T such that σ_t is confined for all $t \geq T$.

Which Vertices Are Unstable At Time t?

- Let
$$\alpha_t = u_0 + \ldots + u_{t-1}$$
be the total number of firings before time t.
- **Lemma**: If σ is confined, then v is unstable at time t if and only if
$$\sigma(v) \equiv -j \pmod{n} \qquad \text{for some } \alpha_{t-1} < j \leq \alpha_t.$$
- Proof uses the fact that for any two vertices v, w, the difference
$$\sigma_t(v) - \sigma_t(w) \mod n$$
doesn't depend on t.

A Recurrence For The Total Activity

- Get a three-term recurrence

$$\alpha_{t+1} = \alpha_t + \sum_{j=\alpha_{t-1}+1}^{\alpha_t} \phi(j)$$

where

$$\phi(j) = \#\{v \,|\, \sigma(v) \equiv -j \pmod{n}\}.$$

- ... which telescopes to a two-term recurrence:

$$\alpha_{t+1} - \alpha_1 = \sum_{s=1}^{t} (\alpha_{s+1} - \alpha_s)$$

$$= \sum_{s=1}^{t} \sum_{j=\alpha_{t-1}+1}^{\alpha_t} \phi(j) = \sum_{j=1}^{\alpha_t} \phi(j).$$

Iterating A Function $\mathbb{N} \to \mathbb{N}$

- $\alpha_{t+1} = f(\alpha_t)$, where
$$f(k) = \alpha_1 + \sum_{j=1}^{k} \phi(j).$$
- Note that
$$\begin{aligned} f(k+n) &= f(k) + \sum_{j=k+1}^{k+n} \phi(j) \\ &= f(k) + \sum_{j=k+1}^{k+n} \#\{v \mid \sigma(v) \equiv -j \pmod{n}\} \\ &= f(k) + n. \end{aligned}$$
- So $f - Id$ is periodic.

Circle Map

- Renormalizing and interpolating

$$g(x) = \frac{(1-\{nx\})f(\lfloor nx \rfloor) + \{nx\}f(\lceil nx \rceil)}{n}$$

yields a continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$g(x+1) = g(x) + 1.$$

- So g descends to a circle map $S^1 \to S^1$ of degree 1.

The Poincaré Rotation Number of a Circle Map

- Suppose $g : \mathbb{R} \to \mathbb{R}$ satisfies $g(x+1) = g(x) + 1$.
- The **rotation number** of g is defined as the limit

$$\rho(g) = \lim_{t\to\infty} \frac{g^t(x)}{t}.$$

- If g is continuous and nondecreasing, then this limit exists and is independent of x.
- If g has a fixed point, then $\rho(g) = ?0$. What about the converse?

Periodic Points and Rotation Number

- More generally, for any rational number p/q

$$\rho(g) = \frac{p}{q} \quad \text{if and only if} \quad g^q - p \text{ has a fixed point.}$$

Chip-Firing Activity and Rotation Number

- We've described how to construct a circle map g from a chip configuration σ.
- **Lemma**: $a(\sigma) = \rho(g)$.
- **Proof**: By construction, $\alpha_t/n = g^t(0)$, so

$$a(\sigma) = \lim_{t\to\infty} \frac{\alpha_t}{nt} = \lim_{t\to\infty} \frac{g^t(0)}{t} = \rho(g).$$

Devil's Staircase Revisited

- Sequence of stable chip configurations $(\sigma_n)_{n\geq 2}$ with σ_n defined on K_n.
- Recall: we assume there is a continuous function $F : [0,1] \to [0,1]$, such that for all $0 \leq x \leq 1$

$$\frac{1}{n}\#\{v \in [n] \,|\, \sigma_n(v) < nx\} \to F(x)$$

as $n \to \infty$.

- Extend F to all of $\mathbb{R}$ by

$$F(x+m) = F(x) + m, \qquad m \in \mathbb{Z},\ x \in [0,1].$$

(Since $F(0) = 0$ and $F(1) = 1$, this extension is continuous.)

Devil's Staircase Revisited

- **Theorem**: For each $y \in [0,1]$

$$s_n(y) \to s(y) := \rho(R_y \circ G) \qquad \text{as } n \to \infty,$$

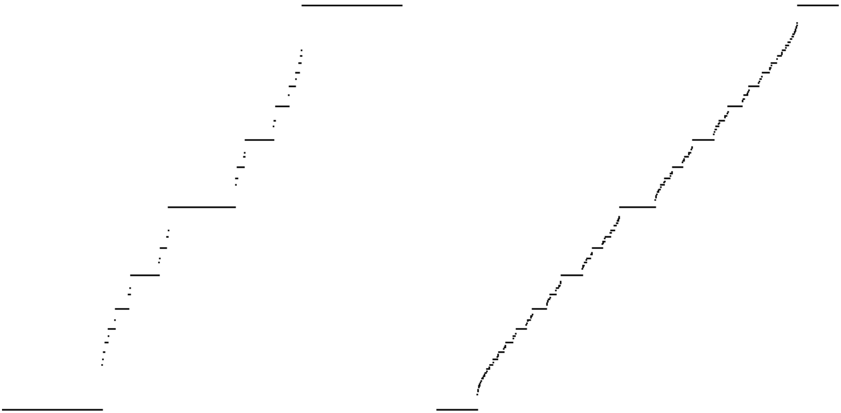
where $G(x) = -F(-x)$, and $R_y(x) = x + y$. Moreover,

  - s is continuous and nondecreasing.
  - If $y \in [0,1]$ is irrational, then $s^{-1}(y)$ is a point.
  - If

$$(\bar{R}_y \circ \bar{G})^q \neq Id : S^1 \to S^1$$

for all $y \in S^1$ and all $q \in \mathbb{N}$, then the fiber $s^{-1}(p/q)$ is an interval of positive length for each rational number $p/q \in [0,1]$.

Different choices of F give different staircases $s(y)$:



Properties of the Rotation Number

- **Continuity**. If $\sup |f_n - f| \to 0$, then $\rho(f_n) \to \rho(f)$.
  $\Rightarrow$ $s_n \to s$, and s is continuous.
- **Monotonicity**. If $f \leq g$, then $\rho(f) \leq \rho(g)$.
  $\Rightarrow$ s is nondecreasing.
- **Instability of an irrational rotation number**. If $\rho(f) \notin \mathbb{Q}$, and $f_1 < f < f_2$, then $\rho(f_1) < \rho(f) < \rho(f_2)$.
  $\Rightarrow$ If $y \notin \mathbb{Q}$, then $s^{-1}(y)$ is a point.

Stability of a rational rotation number

- If $\rho(f) = p/q \in \mathbb{Q}$, and

$$\bar{f}^q \neq Id : S^1 \to S^1$$

then for sufficiently small $\varepsilon > 0$, either

$$\rho(g) = p/q \text{ whenever } f \leq g \leq f + \varepsilon,$$

or

$$\rho(g) = p/q \text{ whenever } f - \varepsilon \leq g \leq f.$$

$\Rightarrow$ The fiber $s^{-1}(p/q)$ is an interval of positive length.

Short Period Attractors

- **Lemma**: If $a(\sigma) = p/q$ in lowest terms, then σ has eventual period q (i.e. $\sigma_{t+q} = \sigma_t$ for all sufficiently large t).
- From the main theorem, it follows that for each $q \in \mathbb{N}$, at least a constant fraction $c_q n$ of the n states $\sigma_n, \sigma_n + 1, \ldots \sigma_n + n - 1$ have eventual period q.
- Curiously, there is also an exclusively period-two window: if the total number of chips is strictly between $n^2 - n$ and n^2, then σ must have eventual period 2.

What About Other Graphs?

- Parallel chip-firing on the torus $\mathbb{Z}/n \times \mathbb{Z}/n$: **F. Bagnoli, F. Cecconi, A. Flammini, A. Vespignani** (*Europhys. Lett.* 2003).
  - Started with $m = \lambda n^2$ chips, each at a uniform random vertex.
  - Ran simulations to find the expected activity as a function of λ.
  - They found a devil's staircase!
- Is there a circle map hiding here somewhere??

Thank You!