Diamond Aggregation

Lionel Levine

Toronto Probability Seminar

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Joint work with Wouter Kager

Talk Outline

- Joint work with Wouter Kager:
  - Internal DLA: from random walk to growth model
  - Uniformly layered walks
  - Limiting shape and fluctuations

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  - Uniformly layered walks
  - Limiting shape and fluctuations
- Joint work with Yuval Peres:
  - Multiple point sources
  - Smash sum of two domains in $\mathbb{R}^d$

From random walk to growth model

Internal DLA

- Given a Markov chain on state space $\mathbb{Z}^2$.
- Start with n particles at the origin.
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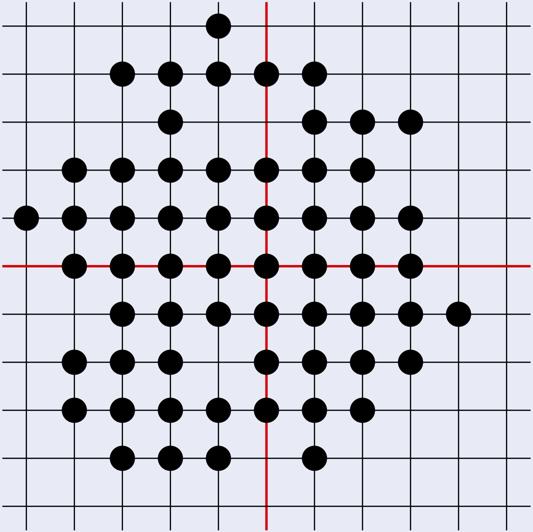
- Let $A(1) = \{o\}$, and

$$A(n+1) = A(n) \cup \{X^n(\tau^n)\}$$

where $X^1, X^2, \ldots$ are independent random walks, and

$$\tau^n = \min\{t \mid X^n(t) \notin A(n)\}.$$

The growth rule illustrated



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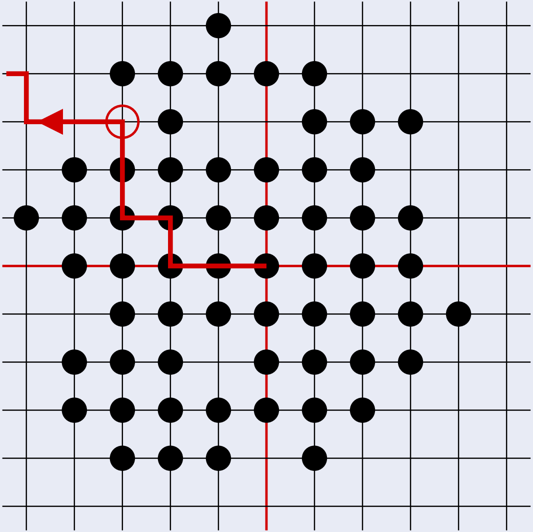

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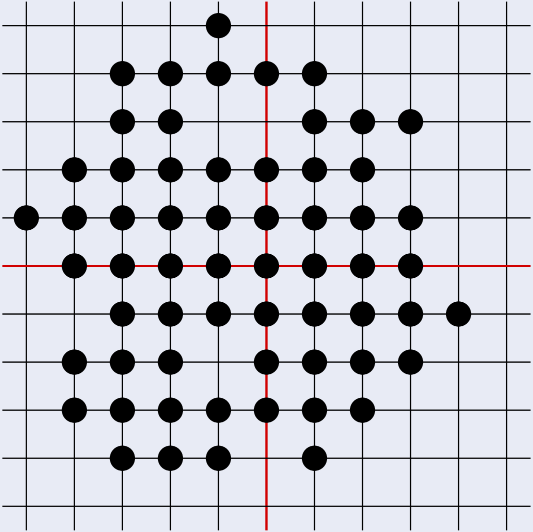
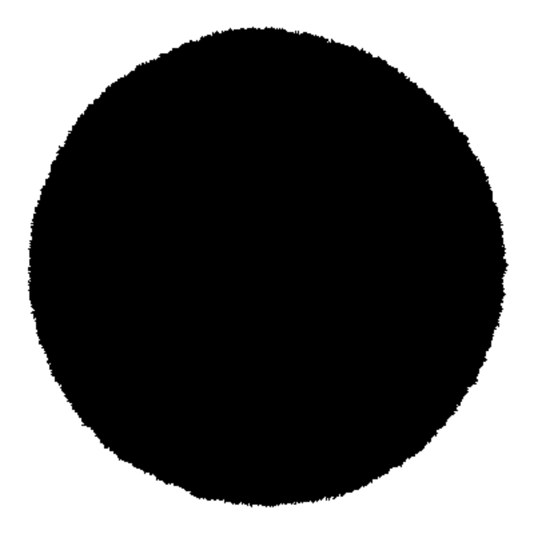


Example: simple random walk



Main questions

1. Limiting shape?

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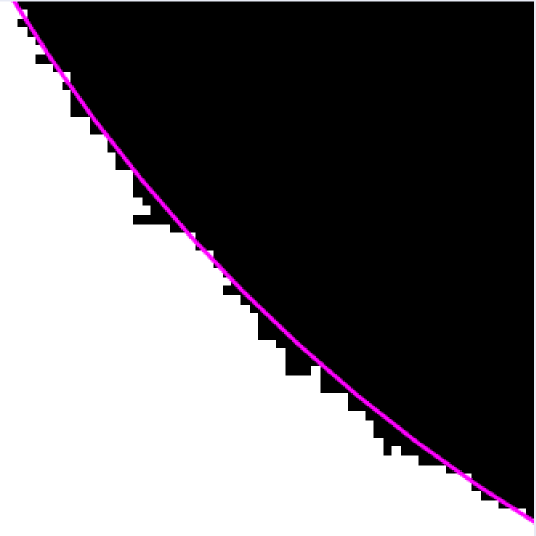

Main questions

1. Limiting shape?
2. Fluctuation size?

Simple random walk

Lawler-Bramson-Griffeath '92

The limiting shape is a disk: $\forall \epsilon > 0$, with probability 1

$$B_{(1-\epsilon)n} \subset A(\pi n^2) \subset B_{(1+\epsilon)n} \quad \text{eventually.}$$

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Strengthened this to show

$$B_{n-f(n)} \subset A(\pi n^2) \subset B_{n+f(n)} \quad \text{eventually}$$

for $f(n) = n^{1/3} \log^4 n$.

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Someone in the audience '09 (?)

The true order of fluctuations $f(n)$ is only logarithmic in n.

What about other walks?

Modify transition probabilities on the axes:

- Steps toward the origin along the x- and y-axes are reflected away from the origin instead. So for $x > 0$,

$$\mathbb{P}((x,0),(x+1,0)) = \frac{1}{2}$$
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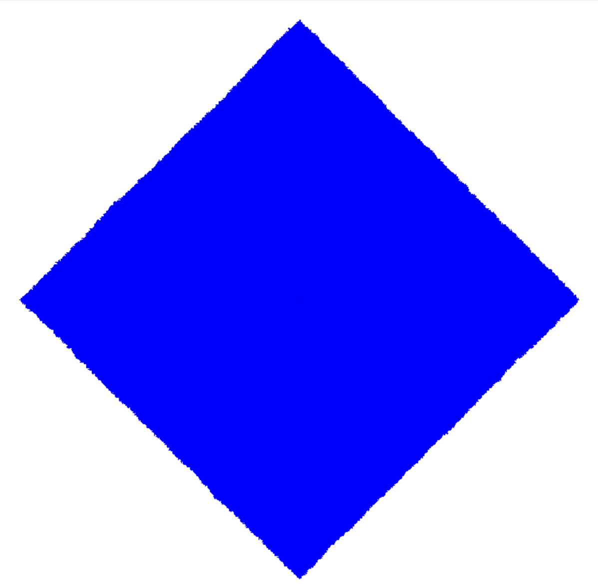
- Off the axes, same as simple random walk.
- Instead of a disk, limiting shape is now a diamond!

Diamond Aggregation

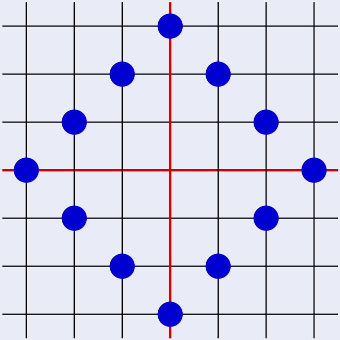


Diamond Layers

Notation

- $\left\|(x,y)\right\| = |x| + |y|.$
- $\mathcal{L}_n =$ the diamond layer of radius n

  $= \left\{ z \in \mathbb{Z}^2 : \|z\| = n \right\}$



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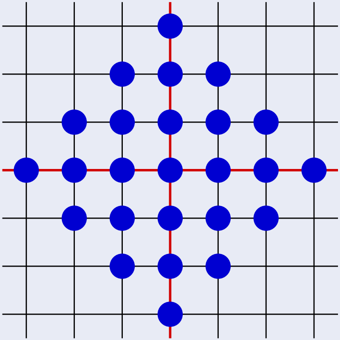
- $\mathcal{D}_n =$ the diamond of radius n
  $= \left\{ z \in \mathbb{Z}^2 : \|z\| \leq n \right\}$
- $d_n = \#\mathcal{D}_n = 2n(n+1) + 1$



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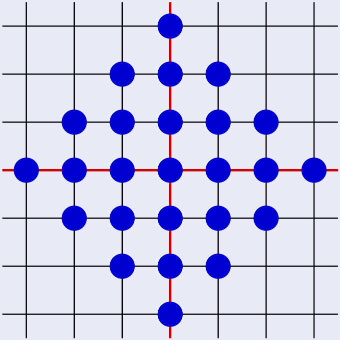

Uniformly layered walk:

Distribution of $X(t)$ is a mixture of uniform distributions on layers $\mathcal{L}_n$.

Uniformly Layered Walk

Discrete time Markov chain $X(t)$ on state space $\mathbb{Z}^2$ satisfying

(U1) $\big| \|X(t+1)\| - \|X(t)\| \big| \leq 1$

(U2) For all $n \geq 1$,

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(U3) For all $k \geq 0$, $n \geq 1$ and all $x \in \mathcal{L}_n$

$$\mathbb{P}_k\Big(X(t) = x \,\Big|\, X(t) \in \mathcal{L}_n\Big) = \frac{1}{4n}$$

where $\mathbb{P}_k$ is the law of the walk started from uniform on layer $\mathcal{L}_k$.

Shape Theorem

Theorem (Kager-L.)

For any uniformly layered walk, with probability 1

$$\mathcal{D}_{n-4\sqrt{n \log n}} \subset A(d_n) \subset \mathcal{D}_{n+20\sqrt{n \log n}} \quad \text{eventually.}$$

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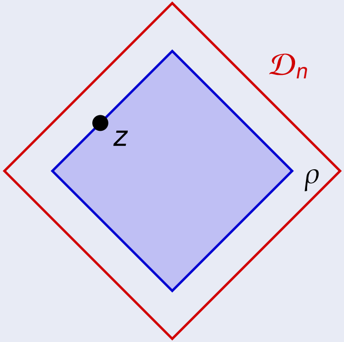
- So all uniformly layered walks have the diamond as their limiting shape.
- Is $\sqrt{n \log n}$ the right order of fluctuations?
- Or do the fluctuations depend on the particular u.l. walk?

Proof sketch: Containing a large diamond

Fix a site $z \in \mathcal{L}_{n-\rho}$. Want an upper bound on $\mathbb{P}(z \notin A(d_n))$.



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- Among the first $d_n - 1$ walks, let

$$M = \#\text{ that first hit } \mathcal{L}_{n-\rho} \text{ at } z.$$
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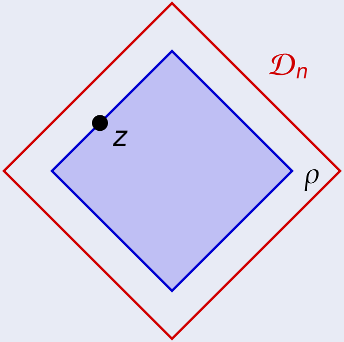
after dropping their particle.



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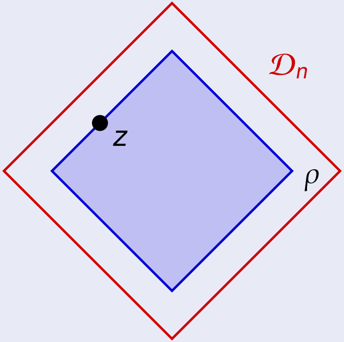
- Then $\{z \notin A(d_n)\} \subset \{L = M\}$.



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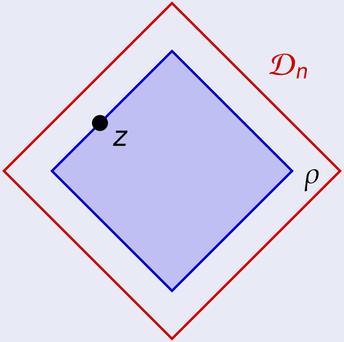


- Both L and M are sums of indicator RV's.
- Main difficulty: The summands of L are dependent.

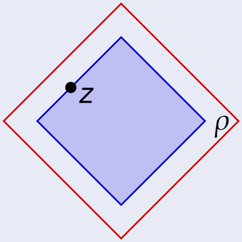
Finding independence

Estimating L

Start one new walk from every site of $\mathcal{D}_{n-\rho-1}$, and let

$$L' = \# \text{ of new walks that first hit } \mathcal{L}_{n-\rho} \text{ at } z.$$

Since at most one particle can attach to the cluster at a given site, $L \leq L'$.



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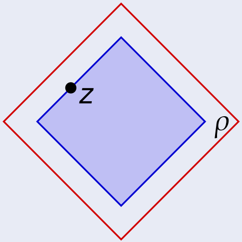

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Strategy

Since both L' and M are sums of independent indicators, show $\mathbb{E}L' < \mathbb{E}M$ and use large deviations to bound $\mathbb{P}(L' \geq M)$.

Separating $\mathbb{E}M$ and $\mathbb{E}L'$

Writing $\ell = n - \rho$, we have

$$\mathbb{E}M = (d_n - 1)\mathbb{P}_o(X(\tau_\ell) = x) = \frac{2n(n+1)}{4\ell} > \frac{n+\rho}{2}.$$

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$$\begin{aligned}\mathbb{E}L' &= \sum_{y \in \mathcal{D}_{\ell-1} - \{o\}} \mathbb{P}_y(X(\tau_\ell) = x) \\ &= \sum_{k=1}^{\ell-1} 4k\mathbb{P}_k(X(\tau_\ell) = x)\end{aligned}$$

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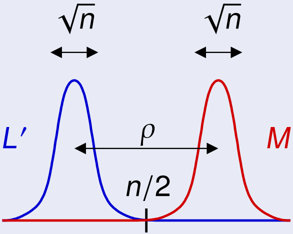
Final step: Concentration



Conclusion

By large deviations for sums of independent indicators,

$$\mathbb{P}(L = M) \leq \mathbb{P}\Big(M \leq \tfrac{n}{2}\Big) + \mathbb{P}\Big(L' \geq \tfrac{n}{2}\Big)$$

has power-law decay for $\rho \sim \sqrt{n \log n}$.

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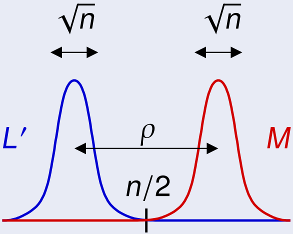

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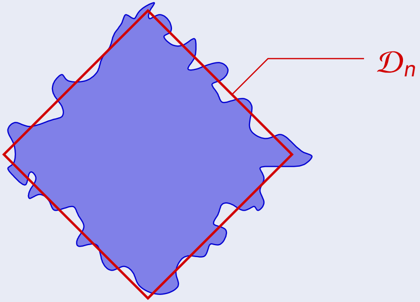
Done by Borel-Cantelli:

$$\sum_{n \ge 1} \sum_{z \in \mathcal{D}_{n-\rho}} \mathbb{P}(z \notin A(d_n)) < \infty.$$

The order of fluctuations



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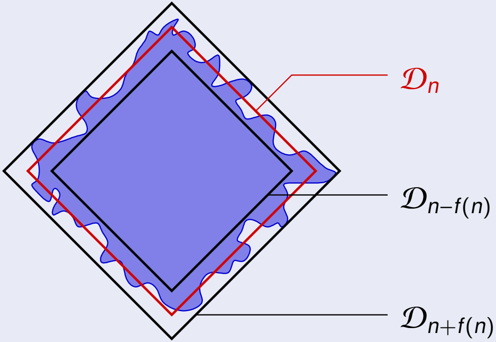
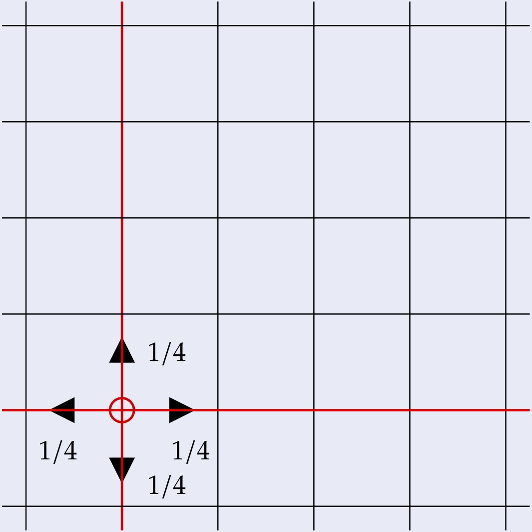


Order of fluctuations:

The slowest rate at which we can let $f(n)$ tend to ∞ so that

$$\mathbb{P}\left(\mathcal{D}_{n-f(n)} \subset A(d_n) \subset \mathcal{D}_{n+f(n)} \text{ eventually}\right) = 1$$

The outward directed layered walk



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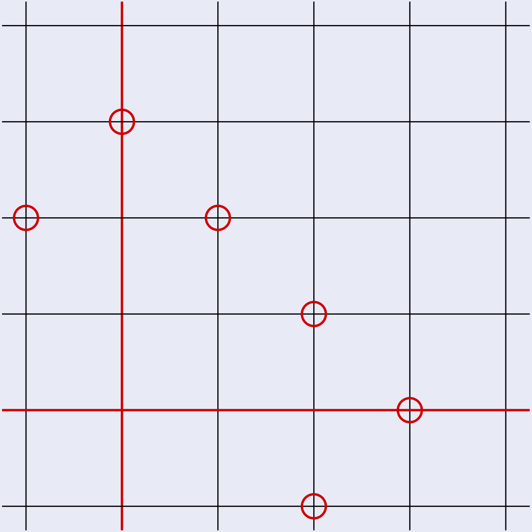

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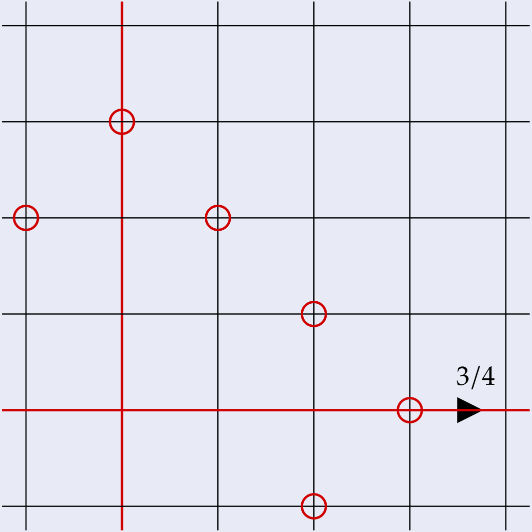

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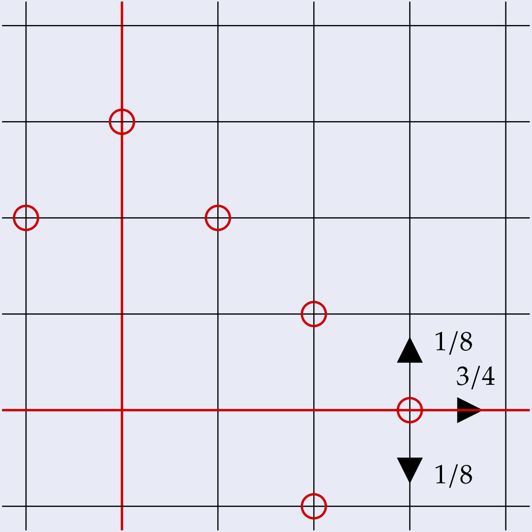

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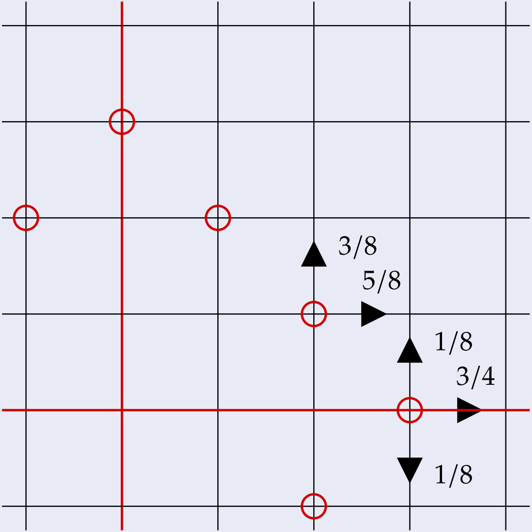

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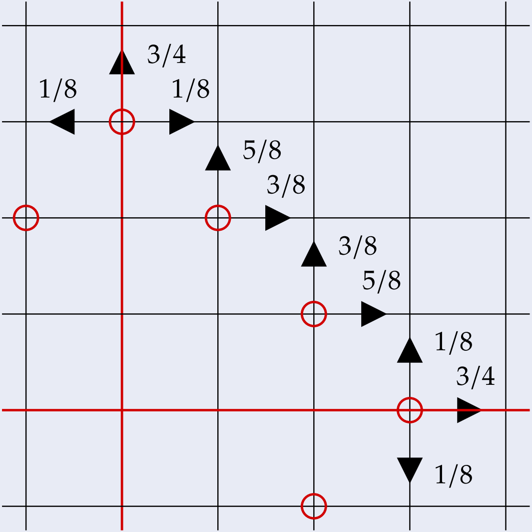
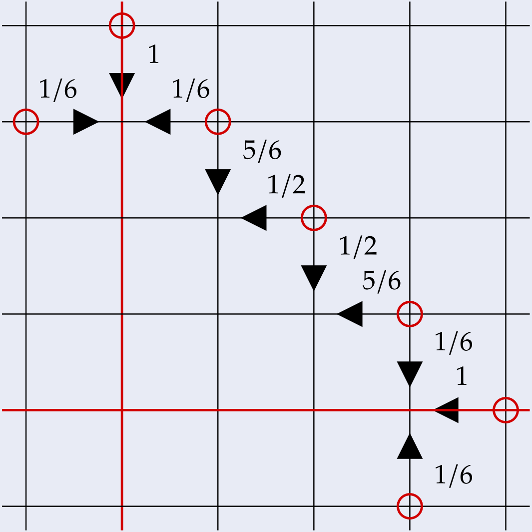


The inward directed layered walk



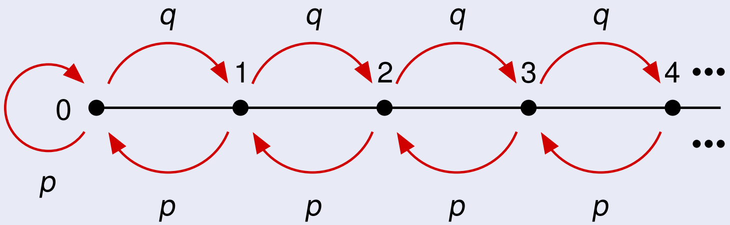
A natural one-parameter family of walks

Mixed transition matrix:

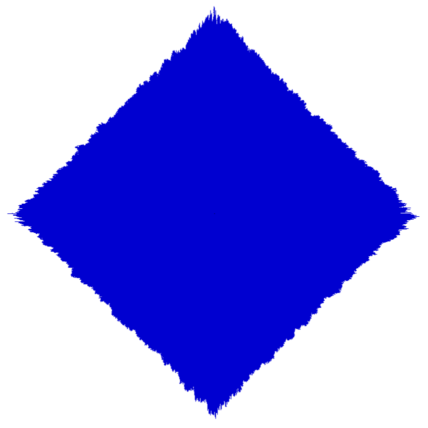
$$Q(x,y) = p\, Q_{\text{in}}(x,y) + q\, Q_{\text{out}}(x,y)$$

where $p \in [0,1)$ and $p + q = 1$.

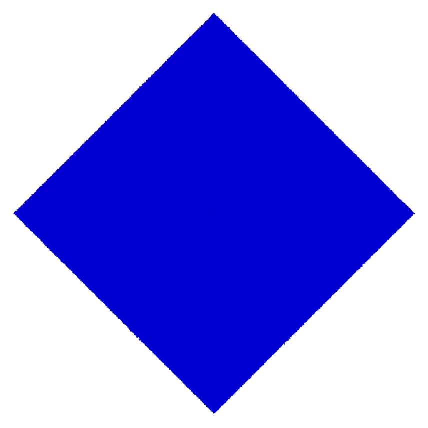
Transitions between layers:



Two diamonds from this family



$p = 0$
walks directed outward



$p = 3/4$
walks biased inward

Logarithmic Fluctuations

Theorem (Kager-L.)

If $p > 1/2$, then with probability 1,

$$\mathcal{D}_{n-6\log_r n} \subset A(d_n) \subset \mathcal{D}_{n+6\log_r n} \quad \text{eventually}$$

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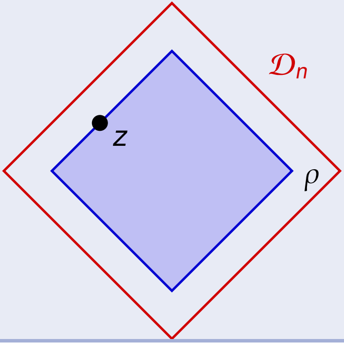
Proof sketch

Fix $z \in \mathcal{L}_{n-\rho}$.

If we stop a walk when it hits $\mathcal{L}_n$, then

$$\mathbb{P}(z \text{ not visited}) \le 4(n-\rho)\left(\frac{q}{p}\right)^{\rho}$$

which has power-law decay for $\rho \sim \log_r n$.



The must-drop-somewhere argument

Conclusion (inner fluctuations)

$$\mathbb{P}\Big(\text{all } d_n \text{ walks visit all sites in } \mathcal{D}_{n-\rho} \text{ before hitting } \mathcal{L}_n\Big) \geq 1 - n^{-2}.$$

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The outer fluctuations

$$\mathbb{P}\Big(\text{all } d_n \text{ walks visit all sites in } \mathcal{D}_n \text{ before hitting } \mathcal{L}_{n+\rho}\Big) \geq 1 - n^{-2}$$

implies

$$\mathbb{P}\Big(A(d_n) \subset \mathcal{D}_{n+\rho}\Big) \geq 1 - n^{-2}$$

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For the outward directed walk ($p = 0$):

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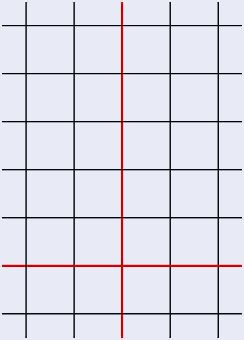
- The X_i are independent geometric RV's.



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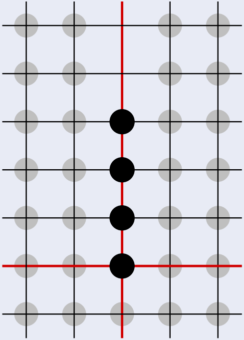
$$T_k = \text{time when } (0, k) \text{ joins the cluster}$$

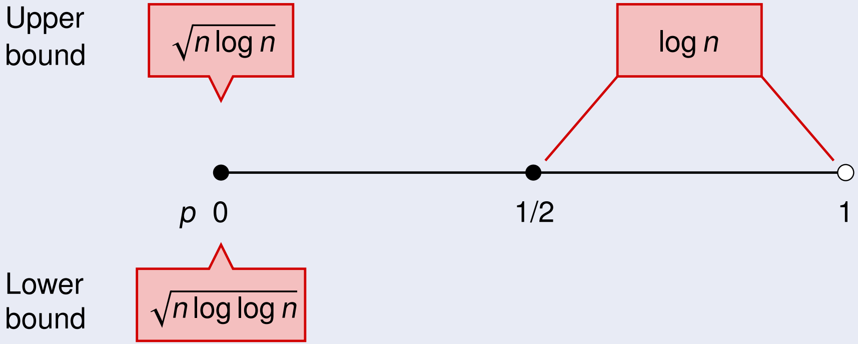
- Set $X_1 = T_1$ and $X_{k+1} = T_{k+1} - T_k$.
- The X_i are independent geometric RV's.

$\Rightarrow$ Law of the iterated logarithm for T_k.

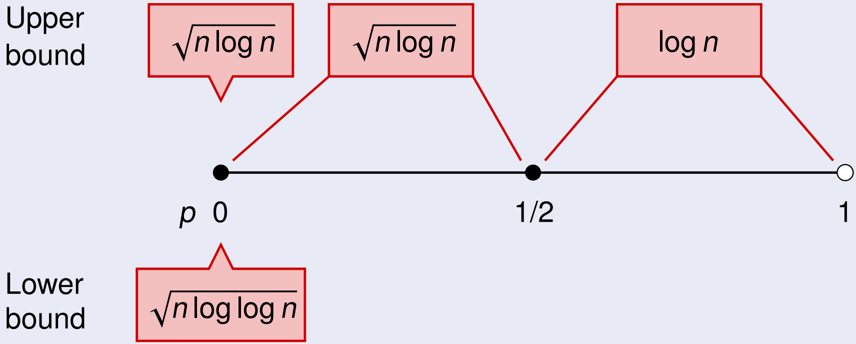
$\therefore$ Fluctuations are at least order $\sqrt{n \log \log n}$.



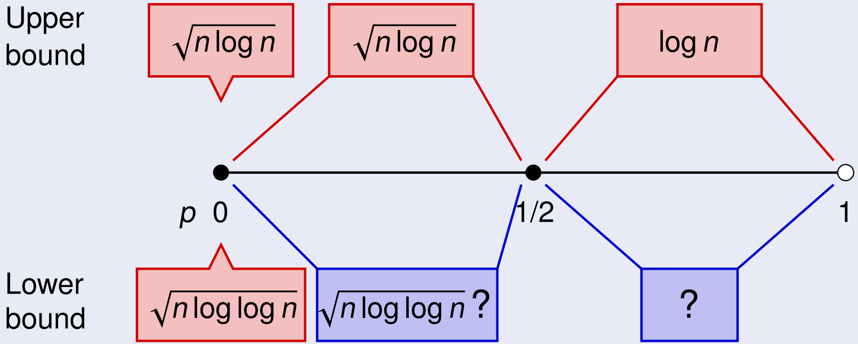
Phase diagram: Order of the fluctuations



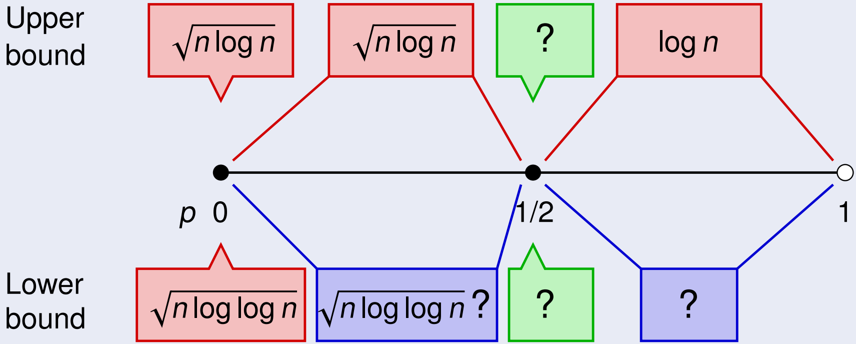
Phase diagram: Order of the fluctuations



Phase diagram: Order of the fluctuations



Phase diagram: Order of the fluctuations



Questions

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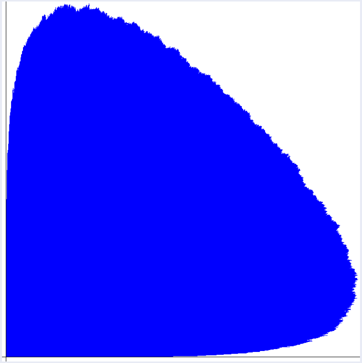
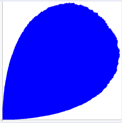
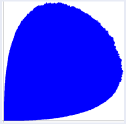
- In $\mathbb{Z}^2$, what if walks start somewhere other than the origin?
- The outward directed layered walk, started from $(1, 1)$:



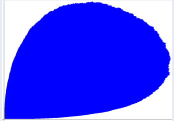
Mystery shapes: Off-center starting point



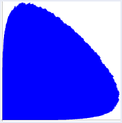
(3,3)



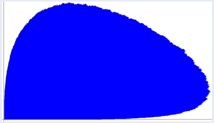
(2,2)



(3,2)



(1,1)



(2,1)

(3,1)