

Escape rates for rotor walks in \mathbb{Z}^d

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Abstract

Rotor walk is a deterministic analogue of random walk. We study its recurrence and transience properties on \mathbb{Z}^d for the initial configuration of all rotors aligned. If n particles in turn perform rotor walks starting from the origin, we show that the number that escape (i.e., never return to the origin) is of order n in dimensions $d \geq 3$, and of order $n/\log n$ in dimension 2.

1 Introduction

In a *rotor walk* on a graph, the successive exits from each vertex follow a prescribed periodic sequence. For instance, in the square grid \mathbb{Z}^2 , successive exits could repeatedly cycle through the sequence North, East, South West. Such walks were first studied in [11] as a model of mobile agents exploring a territory, and in [7] as a model of self-organized criticality. In a lecture at Microsoft in 2003 [8], Jim Propp proposed rotor walk as a deterministic analogue of random walk, which naturally invited the question of whether rotor walk is recurrent in dimension 2 and transient in dimensions 3 and higher. One direction was settled immediately by Oded Schramm, who showed that rotor walk is “at least as recurrent” as random walk. Schramm’s elegant argument, which we recall below, applies to any initial rotor configuration ρ .

The other direction is more subtle because it depends on ρ . We say that ρ is *recurrent* if the rotor walk started at the origin with initial configuration ρ returns to the origin infinitely often; otherwise, we say that ρ is *transient*. Angel and Holroyd [1] showed that for all d there exist initial rotor configurations on \mathbb{Z}^d such that rotor walk is recurrent. These special configurations are primed to send particles initially back toward the origin. The purpose of this note is to analyze the case $\rho = \uparrow$ when all rotors send their first particle in the same direction. To measure how transient this configuration is, we run n rotor walks starting

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from the origin and record whether each returns to the origin or escapes to infinity. We show that the number of escapes is of order n in dimensions $d \geq 3$, and of order $n/\log n$ in dimension 2.

To give the formal definition of a rotor walk, write $\mathcal{E} = \{\pm e_1, \dots, \pm e_d\}$ for the set of $2d$ cardinal directions in \mathbb{Z}^d , and let \mathcal{C} be the set of cyclic permutations of \mathcal{E} . A *rotor mechanism* is a map $m : \mathbb{Z}^d \rightarrow \mathcal{C}$, and a *rotor configuration* is a map $\rho : \mathbb{Z}^d \rightarrow \mathcal{E}$. A *rotor walk* started at x_0 with initial configuration ρ is a sequence of vertices $x_0, x_1, \dots \in \mathbb{Z}^d$ and rotor configurations $\rho = \rho_0, \rho_1, \dots$ such that for all $n \geq 0$

$$x_{n+1} = x_n + \rho_n(x_n).$$

and

$$\rho_{n+1}(x_n) = m(x_n)(\rho_n(x_n))$$

and $\rho_{n+1}(x) = \rho_n(x)$ for all $x \neq x_n$.

For example in \mathbb{Z}^2 , each rotor $\rho(x)$ points North, South, East or West. An example of a rotor mechanism is the permutation North \mapsto East \mapsto South \mapsto West \mapsto North at all $x \in \mathbb{Z}^2$. The resulting rotor walk in \mathbb{Z}^2 has the following description: A particle repeatedly steps in the direction indicated by the rotor at its current location, and then this rotor turns 90 degrees clockwise. Note that this “prospective” convention — move the particle before updating the rotor — differs from the “retrospective” convention of past works such as [1, 3]. In the prospective convention, $\rho(x)$ indicates where the next particle will step from x , instead of where the previous particle stepped. The prospective convention is often more convenient when studying questions of recurrence and transience.

In this paper we fix once and for all a rotor mechanism m on \mathbb{Z}^d . Now depending on the initial rotor configuration ρ , one of two things can happen to a rotor walk started from the origin:

1. The walk eventually returns to the origin; or
2. The walk never returns to the origin, and visits each vertex in \mathbb{Z}^d only finitely often.

Indeed, if any site were visited infinitely often, then each of its neighbors must be visited infinitely often, and so the origin itself would be visited infinitely often. In case 2 we say that the walk “escapes to infinity.” Note that after the walk has either returned to the origin or escaped to infinity, the rotors are in a new configuration.

To quantify the degree of transience of an initial configuration ρ , consider the following experiment: let each of n particles in turn perform rotor walk starting from the origin until either returning to the origin or escaping to infinity. Denote by $I(\rho, n)$ the number of walks that escape to infinity. (Importantly, we do not reset the rotors in between trials!)

Schramm [10] proved that for any ρ ,

$$\limsup_{n \rightarrow \infty} \frac{I(\rho, n)}{n} \leq \alpha_d \tag{1}$$

where α_d is the probability that simple random walk in \mathbb{Z}^d does not return to the origin. Our first result gives a corresponding lower bound for the initial configuration \uparrow in which all rotors start pointing in the same direction: $\uparrow(x) = e_d$ for all $x \in \mathbb{Z}^d$.

Theorem 1. For the rotor walk on \mathbb{Z}^d with $d \geq 3$ with all rotors initially aligned \uparrow , a positive fraction of particles escape to infinity; that is,

$$\liminf_{n \rightarrow \infty} \frac{I(\uparrow, n)}{n} > 0.$$

One cannot hope for such a result to hold for an arbitrary ρ : Angel and Holroyd [1] prove that in all dimensions there exist rotor configurations ρ_{rec} such that $I(\rho_{\text{rec}}, n) = 0$ for all n . Reddy first proposed such a configuration in dimension 3 on the basis of numerical simulations [9].

Our next result concerns the fraction of particles that escape in dimension 2: for any rotor configuration ρ this fraction is at most $\frac{\pi/2}{\log n}$, and for the initial configuration \uparrow it is at least $\frac{c}{\log n}$ for some $c > 0$.

Theorem 2. For rotor walk in \mathbb{Z}^2 with any rotor configuration ρ , we have

$$\limsup_{n \rightarrow \infty} \frac{I(\rho, n)}{n/\log n} \leq \frac{\pi}{2}.$$

Moreover, if all rotors are initially aligned \uparrow , then

$$\liminf_{n \rightarrow \infty} \frac{I(\uparrow, n)}{n/\log n} > 0.$$

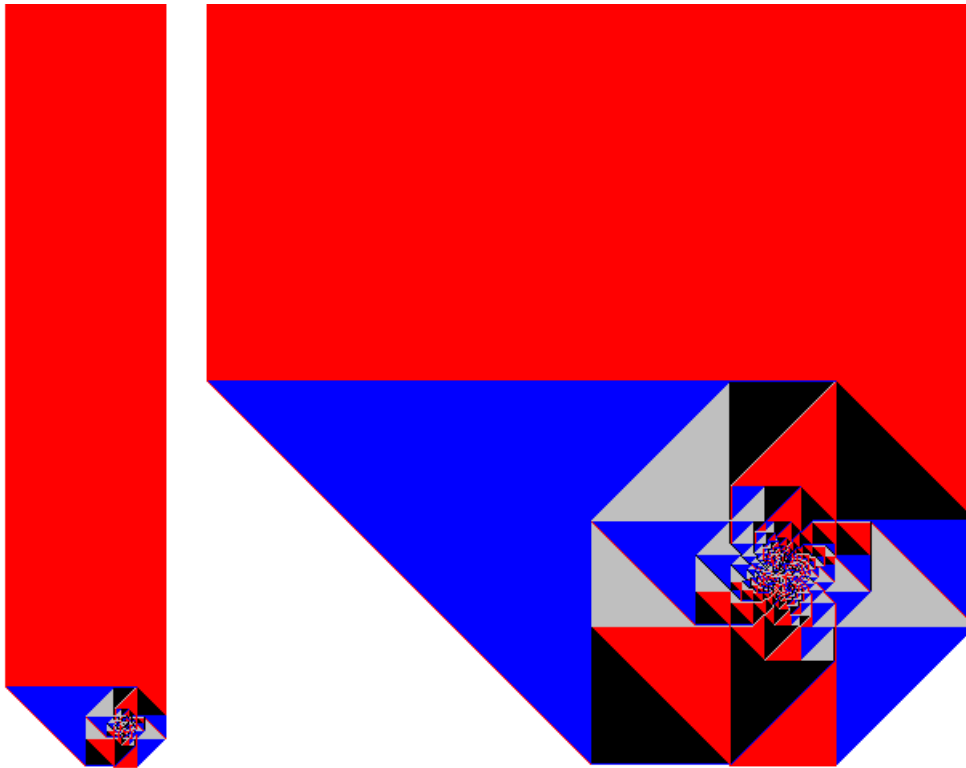


Figure 1: The configuration of rotors in \mathbb{Z}^2 after n particles started at the origin have escaped to infinity, with initial configuration \uparrow (that is, all rotors send their first particle North). Left: $n = 100$; Right: $n = 480$. Each non-white pixel represents a point in \mathbb{Z}^2 that was visited at least once, and its color indicates the direction of its rotor.

2 Schramm's argument

One way to estimate the number of escapes to infinity of a rotor walk is to look at how many particles exit a large ball before returning to the origin. Let

$$\mathcal{B}_r = \{x \in \mathbb{Z}^d : |x| < r\}$$

be the set of lattice points in the open ball of radius r centered at the origin. Here $|x| = (x_1^2 + \dots + x_d^2)^{1/2}$ denotes the Euclidean norm of x . Consider rotor walk started from the origin and stopped on hitting the boundary

$$\partial\mathcal{B}_r = \{y \in \mathbb{Z}^d : y \notin \mathcal{B}_r \text{ and } y \sim x \text{ for some } x \in \mathcal{B}_r\}.$$

Since \mathcal{B}_r is a finite connected graph, this walk stops in finitely many steps.

Starting from initial rotor configuration ρ , let each of n particles in turn perform rotor walk starting from the origin until either returning to the origin or exiting the ball \mathcal{B}_r . Denote by $I_r(\rho, n)$ the number of particles that exit \mathcal{B}_r . The next lemmas give convergence and monotonicity of this quantity.

Lemma 3. [4, Lemma 18] *For any rotor configuration ρ and any $n \in \mathbb{N}$, we have $I_r(\rho, n) \rightarrow I(\rho, n)$ as $r \rightarrow \infty$.*

Proof. Let $w_n(x)$ be the number of exits from x by n rotor walks started at o and stopped if they return to o . Then $I(\rho, n)$ is determined by the values $w_n(x)$ for neighbors x of o .

Let $w_n^r(x)$ be the number of exits from x by n rotor walks started at o and stopped on hitting $\partial\mathcal{B}_r \cup \{o\}$. Then $I_r(\rho, n)$ is determined by the values $w_n^r(x)$ for neighbors x of o .

We first show that $w_n^r \leq w_n$ pointwise. Let $w_n^{r,t}(y)$ be the number of exits from y before time t if the walks are stopped on hitting $\partial\mathcal{B}_r \cup \{o\}$. If $w_n^r \not\leq w_n$, then choose t minimal such that $w_n^{r,t} \not\leq w_n$. Then there is a single point y such that $w_n^{r,t}(y) > w_n(y)$. Note that $y \neq o$, because $w_n^r(o) = w_n(o) = n$. Since $w_n^{r,t}(x) \leq w_n(x)$ for all $x \neq y$, at time t in the finite experiment the site y has received at most as many particles as it ever receives in the infinite experiment. But y has emitted strictly more particles in the finite experiment than it ever emits in the infinite experiment, so the number of particles at y at time t is < 0 , a contradiction.

Now we induct on n to show that $w_n^r \uparrow w_n$ pointwise. Assume that $w_{n-1}^r \uparrow w_{n-1}$. Fix $s > 0$. There exists $R = R(s)$ such that $w_{n-1}^r = w_{n-1}$ on \mathcal{B}_s for all $r \geq R$. If the n th walk returns to o then it does so without exiting \mathcal{B}_s for some S ; in this case $w_n^r = w_n$ on \mathcal{B}_s for all $r \geq \max(R, S)$.

If the n th walk escapes to infinity, then there is some radius S such that after exiting \mathcal{B}_s the walk never returns to \mathcal{B}_s . Now choose R' such that $w_{n-1}^{R'} = w_{n-1}$ on \mathcal{B}_s . Then we claim $w_n^r = w_n$ on \mathcal{B}_s for all $r \geq R'$. Denote by ρ_{n-1}^r (resp. ρ_{n-1}) the rotor configuration after $n-1$ walks started at the origin have stopped on $\partial\mathcal{B}_r \cup \{o\}$ (resp. stopped if they return to o). If $r \geq R'$ then the rotor walks started at o with initial conditions ρ_{n-1}^r and ρ_{n-1} agree until they exit \mathcal{B}_s . Thereafter the latter walk never returns to \mathcal{B}_s , hence $w_n^r \geq w_n$ on \mathcal{B}_s . Since also $w_n^r \leq w_n$ everywhere, the inductive step is complete. \square

For the next lemma we recall the *abelian property* of rotor walk [3, Lemma 3.9]. Let A be a finite subset of \mathbb{Z}^d . In an experiment of the form “run n rotor walks from prescribed starting points until they exit A ,” suppose that we repeatedly choose a particle in A and

ask it to take a rotor walk step. Regardless of our choices, all particles will exit A in finitely many steps; for each $x \in A^c$, the number of particles that stop at x does not depend on the choices; and for each $x \in A$, the number of times we pointed to a particle at x does not depend on the choices.

Lemma 4. [4, Lemma 19] *For any rotor configuration ρ , any $n \in \mathbb{N}$ and any $r < R$, we have $I_R(\rho, n) \leq I_r(\rho, n)$.*

Proof. By the abelian property, we may compute $I_R(\rho, n)$ in two stages. First stop particles when they reach $\partial\mathcal{B}_r \cup \{o\}$, where $o \in \mathbb{Z}^d$ is the origin, and then let the $I_r(\rho, n)$ particles stopped on $\partial\mathcal{B}_r$ continue walking until they reach $\partial\mathcal{B}_R \cup \{o\}$. Therefore at most $I_r(\rho, n)$ particles stop in $\partial\mathcal{B}_R$. \square

Oded Schramm's upper bound (1) begins with the observation that if $2dm$ particles at a single site $x \in \mathbb{Z}^d$ each take a single rotor walk step, the result will be that m particles move to each of the $2d$ neighbors of x . Fix $r, m \in \mathbb{N}$ and consider $N = (2d)^r m$ particles at the origin. Let each particle take a single rotor walk step. Then repeat $r - 1$ times the following operation: let each particle that is not at the origin take a single rotor walk step. The result is that for each path $(\gamma_0, \dots, \gamma_\ell)$ of length $\ell \leq r$ with $\gamma_0 = \gamma_\ell = o$ and $\gamma_i \neq o$ for all $1 \leq i \leq \ell - 1$, exactly $(2d)^{-\ell} N$ particles traverse this path. Denoting the set of such paths by $\Gamma(r)$ and the length of a path γ by $|\gamma|$, the number of particles now at the origin is

$$N \sum_{\gamma \in \Gamma(r)} (2d)^{-|\gamma|} = Np$$

where $p = \mathbb{P}(T_o^+ \leq r)$ is the probability that simple random walk returns to the origin by time r .

Now letting each particle that is not at the origin continue performing rotor walk until hitting $\partial\mathcal{B}_r \cup \{o\}$, the number of particles that stop in $\partial\mathcal{B}_r$ is at most $N(1 - p)$, so

$$\frac{I_r(\rho, N)}{N} \leq 1 - p.$$

This holds for every N which is an integer multiple of $(2d)^r$. For general n , let N be the smallest multiple of $(2d)^r$ that is $\geq n$. Then

$$\frac{I_r(\rho, n)}{n} \leq \frac{I_r(\rho, N)}{N - (2d)^r}$$

The right side is at most $(1 - p)(1 + 2(2d)^r/N)$, so

$$\limsup_{n \rightarrow \infty} \frac{I(\rho, n)}{n} \leq \limsup_{n \rightarrow \infty} \frac{I_r(\rho, n)}{n} \leq 1 - p = \mathbb{P}(T_o^+ > r).$$

As $r \rightarrow \infty$ the right side converges to α_d , completing the proof of (1).

See Holroyd and Propp [4, Theorem 10] for an extension of Schramm's argument to a general irreducible Markov chain with rational transition probabilities.

3 An odometer estimate for balls in all dimensions

To estimate $I_r(\rho, n)$, consider now a slightly different experiment. Let each of n particles started at the origin perform rotor walk until hitting $\partial\mathcal{B}_r$. (The difference is that we do not stop the particles on returning to the origin!) Define the *odometer function* u_n^r by

$$u_n^r(x) = \text{total number of exits from } x \text{ by } n \text{ rotor walks stopped on hitting } \partial\mathcal{B}_r.$$

Note that $u_n^r(x)$ counts the total number of exits (as opposed to the net number).

Now we relate the two experiments.

Lemma 5. *For any $r > 0$ and $n \in \mathbb{N}$ and any initial rotor configuration ρ , we have*

$$I_r(\rho, u_n^r(o)) = n.$$

Proof. Starting with $N = u_n^r(o)$ particles at the origin, consider the following two experiments:

1. Let n of the particles in turn perform rotor walk until hitting $\partial\mathcal{B}_r$.
2. Let N of the particles in turn perform rotor walk until hitting $\partial\mathcal{B}_r \cup \{o\}$.

By the definition of u_n^r , in the first experiment the total number of exits from the origin is exactly N . Therefore the two experiments have exactly the same outcome: n particles reach $\partial\mathcal{B}_r$ and $N - n$ remain at the origin. \square

Our next task is to estimate u_n^r . We begin by introducing some notation. Given a function f on \mathbb{Z}^d , its *gradient* is the function on directed edges given by

$$\nabla f(x, y) := f(y) - f(x).$$

Given a function κ on directed edges of \mathbb{Z}^d , its *divergence* is the function on vertices given by

$$\text{div } \kappa(x) := \frac{1}{2d} \sum_{y \sim x} \kappa(x, y)$$

where the sum is over the $2d$ nearest neighbors of x . The *discrete Laplacian* of f is the function

$$\Delta f(x) := \text{div}(\nabla f)(x) = \frac{1}{2d} \sum_{y \sim x} f(y) - f(x).$$

We recall some results from [6].

Lemma 6. [6, Lemma 5.1] *For a directed edge (x, y) in \mathbb{Z}^d , denote by $\kappa(x, y)$ the net number of crossings from x to y by n rotor walks started at the origin and stopped on exiting \mathcal{B}_r . Then*

$$\nabla u_n^r(x, y) = -2d \kappa(x, y) + R(x, y)$$

for some edge function R satisfying $|R(x, y)| \leq 4d - 2$ for all edges (x, y) .

Denote by $(X_j)_{j \geq 0}$ the simple random walk in \mathbb{Z}^d , whose increments are independent and uniformly distributed on $\mathcal{E} = \{\pm e_1, \dots, \pm e_d\}$. Let $T = \min\{j : X_j \notin \mathcal{B}_r\}$ be the first exit time from the ball of radius r . For $x, y \in \mathcal{B}_r$, let

$$G_r(x, y) = \mathbb{E}_x \#\{j < T | X_j = y\}$$

be the expected number of visits to y by a simple random walk started at x before time T . The following well known estimates can be found in [5, Prop. 1.5.9, Prop. 1.6.7]: for a constant a_d depending only on d ,

$$G_r(x, o) = \begin{cases} a_d(|x|^{2-d} - r^{2-d}) + O(|x|^{1-d}), & d \geq 3 \\ \frac{2}{\pi}(\log r - \log |x|) + O(|x|^{-1}), & d = 2. \end{cases} \quad (2)$$

We will also use [5, Theorem 1.6.6] the fact that in dimension 2,

$$G_r(o, o) = \frac{2}{\pi} \log r + O(1). \quad (3)$$

(As usual, we write $f(n) = \Theta(g(n))$ (respectively, $f(n) = O(g(n))$) to mean that there is a constant $0 < C < \infty$ such that $1/C < f(n)/g(n) < C$ (respectively, $f(n)/g(n) < C$) for all sufficiently large n . Here and in what follows, the constants implied in $O()$ and $\Theta()$ notation depend only on the dimension d .)

The next lemma bounds the L^1 norm of the discrete gradient of the function $G_r(x, \cdot)$. It appears in [6, Lemma 5.6] with the factor of 2 omitted (this factor is needed for x close to the origin). The proof given there actually shows the following.

Lemma 7. *Let $x \in \mathcal{B}_r$ and let $\rho = r + 1 - |x|$. Then for some C depending only on d ,*

$$\sum_{y \in \mathcal{B}_r} \sum_{z \sim y} |G_r(x, y) - G_r(x, z)| \leq C \rho \log \frac{2r}{\rho}.$$

The next lemma is proved in the same way as the inner estimate of [6, Theorem 1.1]. Let $f(x) = nG_r(x, o)$.

Lemma 8. *In \mathbb{Z}^d , let $x \in \mathcal{B}_r$ and $\rho = r + 1 - |x|$. Then,*

$$|u_n^r(x) - f(x)| \leq C \rho \log \frac{2r}{\rho} + 4d.$$

where u_n^r is the odometer function for n particles performing rotor walk stopped on exiting \mathcal{B}_r , and C is the constant in Lemma 7.

Proof. If we consider the rotor walk stopped on exiting \mathcal{B}_r , all sites that have positive odometer value have been hit by particles. Using notation of Lemma 6, we notice that since the net number of particles to enter a site $x \neq o$ not on the boundary is zero, we have $2d \operatorname{div} \kappa(x) = 0$. For the origin, $2d \operatorname{div} \kappa(o) = n$. Also, the odometer function vanishes on the boundary, since the boundary does not emit any particles.

Write $u = u_n^r$. Using the definition of κ in Lemma 6, we see that

$$\Delta u(x) = \operatorname{div} R(x), \quad x \neq o, \quad (4)$$

$$\Delta u(o) = -n + \operatorname{div} R(o). \quad (5)$$

Then $\Delta f(x) = 0$ for $x \in \mathcal{B}_r \setminus \{o\}$ and $\Delta f(o) = -n$ and f vanishes on $\partial\mathcal{B}_r$.

Since $u(X_T)$ is equal to 0, we have

$$u(x) = \sum_{k \geq 0} \mathbb{E}_x(u(X_{k \wedge T}) - u(X_{(k+1) \wedge T})).$$

Also, since the k^{th} term in the sum is zero when $T \leq k$

$$\mathbb{E}_x(u(X_{k \wedge T}) - u(X_{(k+1) \wedge T}) | \mathcal{F}_{k \wedge T}) = -\Delta u(X_k) 1_{\{T > k\}}$$

where $\mathcal{F}_j = \sigma(X_0, \dots, X_j)$ is the standard filtration for the random walk.

Taking expectation of the conditional expectations and using (4) and (5), we get

$$\begin{aligned} u(x) &= \sum_{k \geq 0} \mathbb{E}_x [1_{\{T > k\}} (n 1_{\{X_k = o\}} - \operatorname{div} R(X_k))] \\ &= n \mathbb{E}_x \#\{k < T | X_k = 0\} - \sum_{k \geq 0} \mathbb{E}_x [1_{\{T > k\}} \operatorname{div} R(X_k)]. \end{aligned}$$

So,

$$u(x) - f(x) = -\frac{1}{2d} \sum_{k \geq 0} \mathbb{E}_x \left[1_{\{T > k\}} \sum_{z \sim X_k} R(X_k, z) \right].$$

Let $N(y)$ be the number of edges joining y to $\partial\mathcal{B}_r$. Since $\mathbb{E}_x \sum_{k \geq 0} 1_{\{T > k\}} N(X_k) = 2d$, and $|R| \leq 4d$, the terms with $z \in \partial\mathcal{B}_r$ contribute at most $8d^2$ to the sum. Thus,

$$|u(x) - f(x)| \leq \frac{1}{2d} \left| \sum_{k \geq 0} \mathbb{E}_x \left[\sum_{\substack{y, z \in \mathcal{B}_r \\ y \sim z}} 1_{\{T > k\} \cap \{X_k = y\}} R(y, z) \right] \right| + 4d. \quad (6)$$

Note that for $y \in \mathcal{B}_r$ we have $\{X_k = y\} \cap \{T > k\} = \{X_{k \wedge T} = y\}$. Considering $p_k(y) = \mathbb{P}_x(X_{k \wedge T} = y)$, and noting that R is antisymmetric (because of antisymmetry in Lemma 6), we see that

$$\begin{aligned} \sum_{\substack{y, z \in \mathcal{B}_r \\ y \sim z}} p_k(y) R(y, z) &= - \sum_{\substack{y, z \in \mathcal{B}_r \\ y \sim z}} p_k(z) R(y, z) \\ &= \sum_{\substack{y, z \in \mathcal{B}_r \\ y \sim z}} \frac{p_k(y) - p_k(z)}{2} R(y, z). \end{aligned}$$

Summing over k in (6) and using the fact that $|R| \leq 4d$, we conclude that

$$|u(x) - f(x)| \leq \sum_{\substack{y, z \in \mathcal{B}_r \\ y \sim z}} |G(x, y) - G(x, z)| + 4d.$$

The result now follows from the estimate of the gradient of Green's function in Lemma 7. \square

Now we make our choice of radius, $r = n^{1/(d-1)}$. The next lemma shows that for this value of r , the support of the odometer function contains a large sphere.

Lemma 9. *There exists a sufficiently small $\beta > 0$ depending only on d , such that for any initial rotor configuration and $r = n^{1/(d-1)}$ we have $u_n^r(x) > 0$ for all $x \in \partial\mathcal{B}_{\beta r}$.*

Proof. For $x \in \partial\mathcal{B}_{\beta r}$ we have $\beta r \leq |x| \leq \beta r + 1$. By Lemma 8 we have

$$|u_n^r(x) - f(x)| \leq C'(1 - \beta)r \log \frac{1}{1 - \beta}$$

for a constant C' depending only on d . To lower bound $f(x)$ we use (2): in dimensions $d \geq 3$ we have

$$\begin{aligned} f(x) &= nG_r(x, o) \geq n(a_d(|x|^{2-d} - r^{2-d}) - K|x|^{1-d}) \\ &= a_d(\beta^{2-d} - 1)nr^{2-d} - K\beta^{1-d} \end{aligned}$$

for a constant K depending only on d . Since $r = nr^{2-d}$, we can take $\beta > 0$ sufficiently small so that

$$a_d(\beta^{2-d} - 1)nr^{2-d} - K\beta^{1-d} > 2C'(1 - \beta)r \log \frac{1}{1 - \beta}$$

for all sufficiently large n . Hence $u_n^r(x) > 0$.

In dimension 2, we have $r = n$ and $nG_n(x, o) \geq n\frac{2}{\pi} \log \frac{1}{\beta} - \frac{K}{\beta}$, by (2). So for β small enough, we have that

$$nG_n(x, o) = n\frac{2}{\pi} \log \frac{1}{\beta} - \frac{K}{\beta} > C'(1 - \beta)n \log \frac{1}{1 - \beta}$$

for all sufficiently large n . Hence $u_n^n(x) > 0$. □

Identify \mathbb{Z}^d with $\mathbb{Z}^{d-1} \times \mathbb{Z}$ and call each set of the form $(x_1, \dots, x_{d-1}) \times \mathbb{Z}$ a ‘‘column.’’ Starting n particles at the origin and letting them each perform rotor walk until exiting \mathcal{B}_r where $r = n^{1/(d-1)}$, let $\text{col}(\rho, n)$ be the number of distinct columns that are visited. That is,

$$\text{col}(\rho, n) = \#\{(x_1, \dots, x_{d-1}) : u_n^r(x_1, x_2, \dots, x_d) > 0 \text{ for some } x_d \in \mathbb{Z}\}.$$

By Lemma 9, every site of $\partial\mathcal{B}_{\beta r}$ is visited at least once, so

$$\begin{aligned} \text{col}(\rho, n) &\geq \#\{(x_1, \dots, x_{d-1}) : (x_1, x_2, \dots, x_d) \in \partial\mathcal{B}_{\beta r} \text{ for some } x_d \in \mathbb{Z}\} \\ &\geq C(\beta r)^{d-1} = \Theta(n). \end{aligned} \tag{7}$$

All results so far have not made any assumptions on the initial configuration. The next lemma assumes the initial rotor configuration to be \uparrow . The important property of this initial condition for us is that the first particle to visit a given column travels straight along that column in direction e_d thereafter.

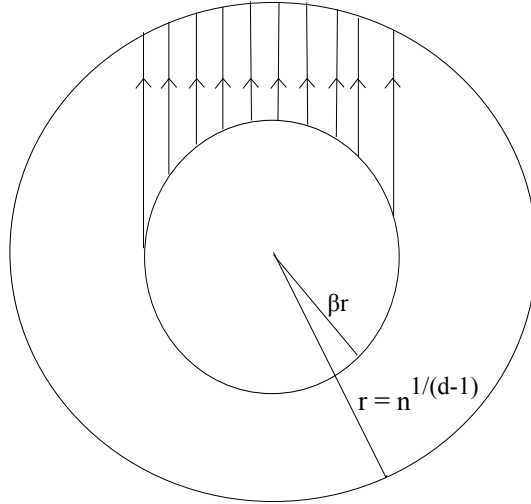


Figure 2: Diagram for the proof of Lemma 10. The first visit to each column results in an escape along that column, so at least $\text{col}(\uparrow, n)$ particles escape.

Lemma 10. *In \mathbb{Z}^d with initial rotor configuration \uparrow , we have*

$$I_R(\uparrow, u_n^r(o)) \geq \text{col}(\uparrow, n)$$

for all $R \geq r$.

Proof. By the abelian property of rotor walk, we may compute $I_R(\rho, u_n^r(o))$ in two stages. First we stop the particles when they first hit $\partial\mathcal{B}_r \cup \{o\}$. Then we let all the particles stopped on $\partial\mathcal{B}_r$ continue walking until they hit $\partial\mathcal{B}_R \cup \{o\}$. By Lemma 5, exactly n particles stop on $\partial\mathcal{B}_r$ during the first stage, and therefore $\text{col}(\uparrow, n)$ distinct columns are visited during the first stage. Because the initial rotors are \uparrow , the first particle to visit a given column travels straight along that column to hit $\partial\mathcal{B}_R$ (Figure 2). Therefore the number of particles stopping in $\partial\mathcal{B}_R$ is at least $\text{col}(\uparrow, n)$. \square

4 The transient case: Proof of Theorem 1

In this section we consider \mathbb{Z}^d for $d \geq 3$. We will prove Theorem 1 by comparing the number of escapes $I(\uparrow, n)$ with $\text{col}(\uparrow, n)$.

Let $r = n^{1/(d-1)}$ and $N = u_n^r(o)$. By the transience of simple random walk in \mathbb{Z}^d for $d \geq 3$ we have

$$f(o) = nG_r(o, o) = \Theta(n).$$

By Lemma 8 we have $|N - f(o)| = O(r)$ and hence $N = \Theta(n)$. By Lemmas 3 and 10 we have $I(\uparrow, N) \geq \text{col}(\uparrow, n)$. Recalling (7) that $\text{col}(\uparrow, n) = \Theta(n)$ and that $I(\uparrow, n)$ is nondecreasing in n , we conclude that there is a constant $c > 0$ depending only on d such that for all sufficiently large n

$$\frac{I(\uparrow, n)}{n} > c$$

which completes the proof.

5 The recurrent case: Proof of Theorem 2

In this section we work in \mathbb{Z}^2 and take $r = n$. We start by estimating the odometer function at the origin for the rotor walk stopped on exiting \mathcal{B}_n .

Lemma 11. *For any initial rotor configuration in \mathbb{Z}^2 we have*

$$u_n^n(o) = \frac{2}{\pi}n \log n + O(n).$$

Proof. By (3), we have $f(o) = nG_n(o, o) = n(\frac{2}{\pi} \log n + O(1))$, and $|u_n^n(o) - f(o)| = O(n)$ by Lemma 8. \square

Turning to the proof of the upper bound in Theorem 2, let $N = u_n^n(o)$. By Lemmas 3 and 4, $I(\rho, N) \leq I_n(\rho, N)$. By Lemma 5, $I_n(\rho, N) = n$. Now by Lemma 11, $\frac{N}{\log N} = \frac{(2/\pi)n \log n + O(n)}{\log n + O(\log \log n)} = (\frac{2}{\pi} + o(1))n$, hence

$$\frac{I(\rho, N)}{N/\log N} \leq \frac{n}{(\frac{2}{\pi} + o(1))n} = \frac{\pi}{2} + o(1).$$

Since $I(\rho, n)$ is nondecreasing in n , the desired upper bound follows.

To show the lower bound for \uparrow we use lemmas 3 and 10 along with (7)

$$I(\uparrow, N) = \lim_{R \rightarrow \infty} I_R(\uparrow, N) \geq \text{col}(\uparrow, n) \geq \beta n = \Theta\left(\frac{N}{\log N}\right).$$

Since $I(\rho, n)$ is nondecreasing in n the desired lower bound follows.

Remark. The proofs of the lower bounds in Theorems 1 and 2 apply to a slightly more general class of rotor configurations than \uparrow . Given a rotor configuration ρ , the *forward path* from x is the path $x = x_0, x_1, x_2, \dots$ defined by $x_{k+1} = x_k + \rho(x_k)$ for $k \geq 0$. Let us say that $x \in \partial\mathcal{B}_r$ has a *simple path to infinity* if the forward path from x is simple (that is, all x_k are distinct) and $x_k \notin \partial\mathcal{B}_r$ for all $k \geq 1$. The proofs we have given for \uparrow remain valid for ρ as long as there is a constant C and a sequence of radii r_1, r_2, \dots with $r_{i+1}/r_i < C$, such that for each i , at least r_i^{d-1}/C sites on $\partial\mathcal{B}_{r_i}$ have disjoint simple paths to infinity. For instance, the rotor configuration

$$\rho(x) = \begin{cases} \alpha, & x_d \geq 0 \\ \beta, & x_d < 0 \end{cases}$$

satisfies this condition as long as $(\alpha, \beta) \neq (-e_d, +e_d)$.

6 Some open questions

We conclude with a few natural questions.

- When is Schramm's bound attained? In \mathbb{Z}^d for $d \geq 3$ with rotors initially aligned in one direction, is the escape rate for rotor walk asymptotically equal to the escape probability of the simple random walk? Theorem 1 shows that the escape rate is positive.

- If random walk on a graph is transient, must there be a rotor configuration ρ for which a positive fraction of particles escape to infinity, that is, $\liminf_{n \rightarrow \infty} \frac{I(\rho, n)}{n} > 0$?
- Let us choose initial rotors $\rho(x)$ for $x \in \mathbb{Z}^d$ independently and uniformly at random from $\{\pm e_1, \dots, \pm e_d\}$. Is the resulting rotor walk recurrent in dimension 2 and transient in dimensions $d \geq 3$? Angel and Holroyd [1, Corollary 6] prove that two initial configurations differing in only a finite number of rotors are either both recurrent or both transient. Hence the set of recurrent ρ is a tail event and consequently has probability 0 or 1.
- Starting from initial rotor configuration \uparrow in \mathbb{Z}^2 , let ρ_n be the rotor configuration after n particles have escaped to infinity. Does $\rho_n(nx, ny)$ have a limit as $n \rightarrow \infty$? Figure 1 suggests that the answer is yes.
- Consider rotor walk in \mathbb{Z}^2 with a drift to the north: each rotor mechanism is period 5 with successive exits cycling through North, North, East, South, West. Is this walk transient for all initial rotor configurations?

Angel and Holroyd resolved many of these questions when \mathbb{Z}^d is replaced by an arbitrary rooted tree: if only finitely many rotors start pointing toward the root (recall we use the prospective convention), then the escape rate for rotor walk started at the root equals the escape probability \mathcal{E} for random walk started at the root [2, Theorem 3]. On the other hand if *all* rotors start pointing toward the root, then the rotor walk is recurrent [2, Theorem 2(iii)]. On the regular b -ary tree, the i.i.d. uniformly random initial rotor configuration has escape rate $\mathcal{E} = 1/b$ for $b \geq 3$ but is recurrent for $b = 2$ [2, Theorem 6]. In the latter case particles travel extremely far [2, Theorem 7]: There is a constant $c > 0$ such that with probability tending to 1 as $n \rightarrow \infty$, one of the first n particles reaches distance e^{cn} from the root before returning!

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