

# An algebraic analogue of a formula of Knuth

Lionel Levine

(MIT)

FPSAC, San Francisco,

August 6, 2010

## Talk Outline

- ▶ Knuth's formula: generalizing  $n^{n-1}$ .
- ▶ ... with weights: generalizing  $(x_1 + \dots + x_n)^{n-1}$ .
- ▶ ... with group structure: generalizing  $(\mathbb{Z}/n\mathbb{Z})^{n-1}$ .
- ▶ Recent developments!

## Starting Point: Cayley's Theorem

- ▶ The number of rooted trees on  $n$  labeled vertices is  $n^{n-1}$ .
- ▶ Refinement: The number of trees with **degree sequence**  $(d_1, \dots, d_n)$  is the coefficient of  $x_1^{d_1} \dots x_n^{d_n}$  in

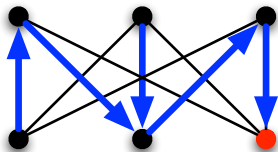
$$nx_1 \dots x_n (x_1 + \dots + x_n)^{n-2}.$$

- ▶ We can break this out by root:

$$\sum_{r=1}^n \prod_{i \neq r} x_i \cdot x_r (x_1 + \dots + x_n)^{n-2}$$

**outdegrees**                      **indegrees**

## Oriented Spanning Trees



An oriented spanning tree of  $K_{3,3}$ .

- ▶ Let  $G = (V, E)$  be a finite directed graph.
- ▶ An *oriented spanning tree* of  $G$  is a subgraph  $T = (V, E')$  such that
  - ▶ one vertex, the **root**, has outdegree 0;
  - ▶ all other vertices have outdegree 1;
  - ▶  $T$  has **no oriented cycles**  $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k \rightarrow v_1$ .

## Complexity of A Directed Graph

- ▶ The number

$$\kappa(G) = \# \text{ of oriented spanning trees of } G$$

is sometimes called the *complexity* of  $G$ .

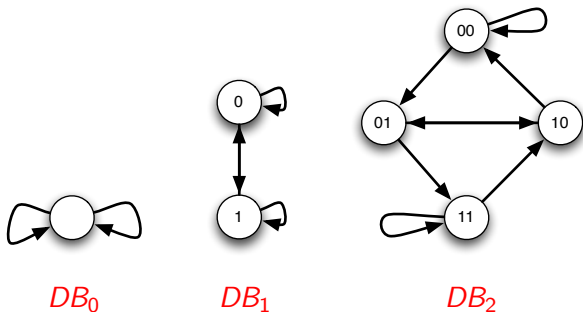
- ▶ Examples:

$$\kappa(K_n) = n^{n-1}$$

$$\kappa(K_{m,n}) = (m+n)m^{n-1}n^{m-1}$$

$$\kappa(DB_n) = 2^{2^n-1}$$

## The De Bruijn Graph $DB_n$



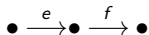
- ▶ vertices  $\{0,1\}^n$ , edges  $\{0,1\}^{n+1}$ .
- ▶ The endpoints of the edge  $e = b_1 \dots b_{n+1}$  are its prefix and suffix:

$$b_1 \dots b_n \xrightarrow{e} b_2 \dots b_{n+1}.$$

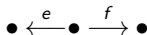
## Directed Line Graphs

- ▶  $G = (V, E)$  : finite directed graph
- ▶  $s, t : E \rightarrow V$
- ▶ Edge  $e$  is directed like this:  $s(e) \xrightarrow{e} t(e)$
- ▶ The *directed line graph*  $\mathcal{L}G = (E, E_2)$  of  $G$  has
  - ▶ Vertex set  $E$ , the edge set of  $G$ .
  - ▶ Edge set

$$E_2 = \{(e, f) \in E \times E \mid s(f) = t(e)\}.$$



$$(e, f) \in E_2$$

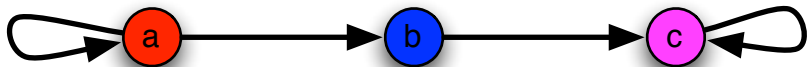
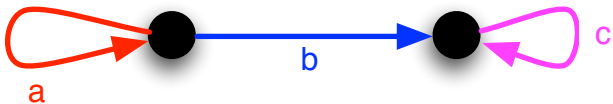


$$(e, f) \notin E_2$$



$$(e, f) \notin E_2$$

## A Graph $G$ and Its Directed Line Graph $\mathcal{L}G$





## Examples of Directed Line Graphs

- ▶  $\vec{K}_n = \mathcal{L}(\text{one vertex with } n \text{ loops})$ .
- ▶  $\vec{K}_{m,n} = \mathcal{L}(\text{two vertices } \{a, b\} \text{ with } m \text{ edges } a \rightarrow b \text{ and } n \text{ edges } b \rightarrow a)$ .
- ▶  $DB_n = \mathcal{L}(DB_{n-1})$ .
- ▶ Iterated line graphs:  $\mathcal{L}^n G = (E_n, E_{n+1})$ , where

$$E_n = \{\text{directed paths of } n \text{ edges in } G\}.$$

## Spanning Tree Enumerators

- ▶ Let  $(x_v)_{v \in V}$  and  $(x_e)_{e \in E}$  be indeterminates, and let

$$\kappa^{\text{edge}}(G, \mathbf{x}) = \sum_T \prod_{e \in T} x_e$$

$$\kappa^{\text{vertex}}(G, \mathbf{x}) = \sum_T \prod_{e \in T} x_{t(e)}$$

The sums are over all oriented spanning trees  $T$  of  $G$ .

- ▶ Example:

$$\kappa^{\text{vertex}}(K_n, \mathbf{x}) = (x_1 + \cdots + x_n)^{n-1}.$$

## Knuth's Formula

- ▶  $G = (V, E)$  : finite directed graph with no sources
- ▶ outdegrees  $a_1, \dots, a_n$
- ▶ indegrees  $b_1, \dots, b_n \geq 1$
- ▶  $\mathcal{L}G$  : the directed line graph of  $G$
- ▶ **Theorem (Knuth, 1967)**. For any edge  $e_*$  of  $G$ ,

$$\kappa(\mathcal{L}G, e_*) = \alpha(G, e_*) \prod_{i=1}^n a_i^{b_i-1}$$

where

$$\alpha(G, e_*) = \kappa(G, t(e_*)) - \frac{1}{a_*} \sum_{\substack{t(e)=t(e_*) \\ e \neq e_*}} \kappa(G, s(e)).$$

and  $a_*$  is the outdegree of  $t(e_*)$ .

## Weighted Knuth's Formula

- ▶  $G$  : finite directed graph with no sources
- ▶  $\mathcal{L}G$  : its directed line graph
- ▶  $b_1, \dots, b_n \geq 1$  : the indegrees of  $G$ .
- ▶ **Theorem (L.)**

$$\kappa^{\text{vertex}}(\mathcal{L}G, \mathbf{x}) = \kappa^{\text{edge}}(G, \mathbf{x}) \prod_{i \in V} \left( \sum_{s(e)=i} x_e \right)^{b_i-1}.$$

- ▶ Both sides are polynomials in the edge variables  $x_e$ .

## Specializing $x_e = 1$

- ▶ Complexity of a line graph:

$$\kappa(\mathcal{L}G) = \kappa(G) \prod_{i=1}^n a_i^{b_i-1}.$$

- ▶ Examples:

- ▶  $G =$  one vertex with  $n$  loops,  $\mathcal{L}G = K_n$ , get  $n^{n-1}$ .
- ▶  $G =$  two vertices,  $\mathcal{L}G = K_{m,n}$ , get  $(m+n)m^{n-1}n^{m-1}$ .
- ▶  $G = DB_{n-1}$ ,  $\mathcal{L}G = DB_n$ :

$$\begin{aligned}\kappa(DB_n) &= \kappa(DB_{n-1}) \cdot 2^{2^{n-1}} \\ &= \kappa(DB_{n-2}) \cdot 2^{2^{n-1}} \cdot 2^{2^{n-2}} \\ &= \dots \\ &= 2^{2^n-1}.\end{aligned}$$

## Rooted Version

- ▶ Fix an edge  $e_* = (w_*, v_*)$  of  $G$ .
- ▶ Let  $b_*$  be the indegree of  $v_*$ .
- ▶ **Theorem** (L.) If  $b_i \geq 1$  for all  $i$ , and  $b_* \geq 2$ , then

$$\kappa^{\text{vertex}}(\mathcal{L}G, e_*, \mathbf{x}) = x_{e_*} \kappa^{\text{edge}}(G, w_*, \mathbf{x}) \frac{\prod_{i \in V} (\sum_{s(e)=i} x_e)^{b_i-1}}{\sum_{s(e)=v_*} x_e}.$$

## Matrix-Tree Theorem

$$\kappa^{edge}(G, \mathbf{x}) = [t] \det(t \cdot Id - \Delta^{edge}).$$

$$\kappa^{vertex}(G, \mathbf{x}) = [t] \det(t \cdot Id - \Delta^{vertex}).$$

- ▶ Goal: relate  $\Delta_G^{edge}$  with  $\Delta_{LG}^{vertex}$ .

## The Missing Link: Directed Incidence Matrices

- Consider the  $K$ -linear maps

$$\begin{aligned} A: K^E &\rightarrow K^V, & B: K^V &\rightarrow K^E \\ e &\mapsto t(e) & v &\mapsto \sum_{s(e)=v} x_e e. \end{aligned}$$

Then

$$\begin{aligned} \Delta_G^{edge} &= AB - D \\ \Delta_{LG}^{vertex} &= BA - D^L \end{aligned}$$

where  $D$  and  $D^L$  are the diagonal matrices

$$D(v) = \left( \sum_{s(e)=v} x_e \right) v, \quad D^L(e) = \left( \sum_{s(f)=t(e)} x_f \right) e.$$



## Intertwining $\Delta_G^{edge}$ and $\Delta_{\mathcal{L}G}^{vertex}$

$$A\Delta_{\mathcal{L}G}^{vertex} = A(BA - D^{\mathcal{L}}) = ABA - DA = (AB - D)A = \Delta_G^{edge} A$$

- ▶ In particular

$$\Delta_{\mathcal{L}G}^{vertex}(\ker A) \subset \ker A.$$

- ▶ Writing  $K^E = \ker A \oplus \text{Im}(A^T)$  puts  $\Delta_{\mathcal{L}G}^{vertex}$  in block triangular form.

## The Proof Falls Into Place

- ▶ Since  $G$  has no sources,  $A : K^E \rightarrow K^V$  is onto.
  - ▶ So  $AA^T$  has full rank.
  - ▶ So  $A : \text{Im}(A^T) \rightarrow K^V$  is an isomorphism.
  - ▶ So  $\det(\Delta_{LG}^{\text{vertex}}|_{\text{Im}(A^T)}) = \det \Delta_G^{\text{edge}} = \kappa(G, \mathbf{x})$ .
- ▶ Eigenvalues of  $\Delta_{LG}^{\text{vertex}}|_{\ker A}$  are  $\sum_{s(e)=i} x_e$ , each with multiplicity  $b_i - 1$ .  $\square$

## Comparison with Knuth

- ▶ Knuth's formula involved the strange quantity

$$\alpha(G, e_*) = \kappa(G, t(e_*)) - \frac{1}{a_*} \sum_{\substack{t(e)=t(e_*) \\ e \neq e_*}} \kappa(G, s(e)).$$

- ▶ Why is it missing from our formulas?

## The Unicycle Lemma

- ▶ A **unicycle** of  $G$  is an oriented spanning tree together with an outgoing edge from the root.
- ▶ By counting unicycles through  $v_*$  in two ways, we get:
- ▶ **Lemma.**

$$\kappa^{edge}(G, v_*, \mathbf{x}) \sum_{s(e)=v_*} x_e = \sum_{t(e)=v_*} \kappa^{edge}(G, s(e), \mathbf{x}) x_e.$$

- ▶ So Knuth's formula simplifies to

$$\kappa(\mathcal{L}G, e_*) = \frac{1}{a_*} \kappa(G, s(e_*)) \prod_{i=1}^n a_i^{b_i-1}.$$

## The Sandpile Group of a Graph

- ▶  $K(G, v_*) \simeq \mathbb{Z}^{n-1} / \Delta \mathbb{Z}^{n-1}$ , where

$$\Delta = D - A$$

is the **reduced Laplacian** of  $G$ .

- ▶ Lorenzini '89/'91 (“group of components”), Dhar '90, Biggs '99 (“critical group”), Baker-Norine '07 (“Jacobian”).
  - ▶ Directed graphs: Holroyd et al. '08
- ▶ **Matrix-tree theorem**:

$$\#K(G, v_*) = \det \Delta = \#\{\text{spanning trees of } G \text{ rooted at } v_*\}.$$

- ▶ Choice of sink:  $K(G, v_*) \simeq K(G, v'_*)$  if  $G$  is **Eulerian**.

## Maps Between Sandpile Groups

**Theorem (L.)** If  $G$  is Eulerian, then the map

$$\begin{aligned}\mathbb{Z}^E &\rightarrow \mathbb{Z}^V \\ e &\mapsto \mathfrak{t}(e)\end{aligned}$$

descends to a surjective group homomorphism

$$K(\mathcal{L}G, e_*) \rightarrow K(G, \mathfrak{t}(e_*)).$$

## Maps Between Sandpile Groups

**Theorem (L.)** If  $G$  is **balanced  $k$ -regular**, then the map

$$\begin{aligned} \mathbb{Z}^V &\rightarrow \mathbb{Z}^E \\ v &\mapsto \sum_{s(e)=v} e \end{aligned}$$

descends to an isomorphism of groups

$$K(G) \simeq kK(\mathcal{L}G).$$

- ▶ Analogous to results of **Berget, Manion, Maxwell, Potechin and Reiner** on undirected line graphs. [arXiv:0904.1246](https://arxiv.org/abs/0904.1246)

## The Sandpile Group of $DB_n$

- ▶ De Bruijn Graph  $DB_n = \mathcal{L}^n$  (a single vertex with 2 loops).
- ▶ **Theorem** (L.)

$$K(DB_n) = \bigoplus_{j=1}^{n-1} (\mathbb{Z}/2^j\mathbb{Z})^{2^{n-1-j}}.$$

- ▶ Generalized by **Bidkhor** and **Kishore** to  $k$ -ary De Bruijn graphs for any  $k$ .



## Equating Exponents

- ▶ By counting spanning trees, we know that

$$\#K(DB_n) = \kappa(DB_n, v_*) = 2^{2^n - n - 1}.$$

- ▶ Now write

$$K(DB_n) = \mathbb{Z}_2^{a_1} \oplus \mathbb{Z}_4^{a_2} \oplus \mathbb{Z}_8^{a_3} \oplus \dots \oplus \mathbb{Z}_{2^m}^{a_m}$$

for some nonnegative integers  $m$  and  $a_1, \dots, a_m$  satisfying

$$\sum_{j=1}^m j a_j = 2^n - n - 1. \quad (1)$$

- ▶ By the previous theorem and inductive hypothesis

$$K(DB_{n-1}) \simeq 2K(DB_n)$$
$$\mathbb{Z}_2^{2^{n-3}} \oplus \mathbb{Z}_4^{2^{n-4}} \oplus \dots \oplus \mathbb{Z}_{2^{n-2}} \simeq \mathbb{Z}_2^{a_2} \oplus \mathbb{Z}_4^{a_3} \oplus \dots \oplus \mathbb{Z}_{2^{m-1}}^{a_m}.$$

- ▶ So  $m = n - 1$  and  $a_j = 2^{n-j-1}$ .  $\square$

## A (Formerly) Open Problem From EC1

- ▶ In EC1, Stanley asks for a bijection

{pairs of binary De Bruijn sequences of order  $n$ }



{all binary sequences of length  $2^n$ }

- ▶ Both sets have cardinality  $2^{2^n}$ .

## Recent Developments

- ▶ In [arXiv:0910.3442](https://arxiv.org/abs/0910.3442), **Bidkhor** and **Kishore** give a bijective proof of the weighted Knuth formula

$$\kappa^{\text{vertex}}(\mathcal{L}G, \mathbf{x}) = \kappa^{\text{edge}}(G, \mathbf{x}) \prod_{i \in V} \left( \sum_{s(e)=i} x_e \right)^{b_i-1}$$

and use it to solve Stanley's problem!

- ▶ **Perkinson, Salter and Xu** give a surjective map

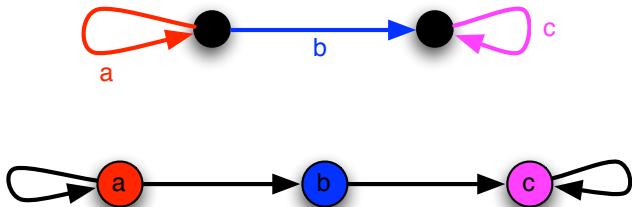
$$K(\mathcal{L}G, e_*) \rightarrow K(G, s(e_*))$$

even when  $G$  is not Eulerian.

## Now What?

	unweighted	weighted
enumeration	$\kappa(G)$	$\kappa(G, \mathbf{x})$
algebra	$K(G)$	?

# Thank You!



## References:

- ▶ D. E. Knuth, Oriented subtrees of an arc digraph, *J. Comb. Theory* **3** (1967), 309–314.
- ▶ L., Sandpile groups and spanning trees of directed line graphs, *J. Comb. Theory A*, to appear. [arXiv:0906.2809](https://arxiv.org/abs/0906.2809)