The gap of the area-weighted Motzkin spin chain is exponentially small

Lionel Levine[1, *] and Ramis Movassagh[2, †]

^{1}Department of Mathematics, Cornell University, Ithaca, NY 14853

^{2}Department of Mathematical Sciences, IBM T. J . Watson Research Center, Yorktown Heights, NY 10598

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We prove that the energy gap of the model proposed by Zhang, Ahmadain, and Klich [1] is exponentially small in the square of the system size. In [2] a class of exactly solvable quantum spin chain models was proposed that have integer spins (s), with a nearest neighbors Hamiltonian, and a unique ground state. The ground state can be seen as a uniform superposition of all $s-$colored Motzkin walks. The half-chain entanglement entropy provably violates the area law by a square root factor in the system's size ($\sim \sqrt{n}$) for $s > 1$. For $s = 1$, the violation is logarithmic [3]. Moreover in [2] it was proved that the gap vanishes polynomially and is $O(n^{-c})$ with $c \geq 2$.

Recently, a deformation of [2], which we call "weighted Motzkin quantum spin chain" was proposed [1]. This model has a unique ground state that is a superposition of the $s-$colored Motzkin walks weighted by $t^{\text{area\{Motzkin walk\}}}$ with $t > 1$. The most surprising feature of this model is that it violates the area law by a factor of n. Here we prove that the gap of this model is upper bounded by $8ns \, t^{-n^2/3}$ for $t > 1$.

Contents

I. CONTEXT AND SUMMARY OF THE RESULTS

In recent years there has been a surge of activities in developing new exactly solvable models that give large violations of the area law for the entanglement entropy [1–5]. The notion of exactly solvable in these works means that the ground state can be written down analytically and the gap to the first excited state can be quantified. In some cases certain correlation functions can be analytically calculated as well (e.g., [6]). Understanding the gap is important for the physics of quantum many-body systems.

Area law says that the entanglement entropy of the ground state of a gapped Hamiltonian between a subsystem and the rest scales as the boundary of the subsystem. This has only rigorously been proved in one dimension [7], yet is believed to hold in higher dimensions as well. For gapless one-dimensional systems, based on detailed and precise results in critical systems described by conformal field theories, the area law was believed to be violated by at most a logarithmic factor in the system's size. The above presume physical reasonability of the underlying model, which means the Hamiltonian is local, translationally invariant in the bulk with a unique ground state.

In [2] a class of exactly solvable quantum spin-chains was proposed that violate the area law by a square root factor in the system's size. They have positive integer spins ($s > 1$), the Hamiltonian is nearest neighbors with a unique ground state that can be seen as a uniform superposition of $s-$colored Motzkin walks. The half-chain entanglement entropy provably scales as a square root factor in the system's size ($\sim \sqrt{n}$). The power-law violation of the entanglement entropy in that work provides a counter-example to the widely believed notion, that translationally invariant spin chains with a unique ground state and local interactions can violate the area law by at most a logarithmic factor in the system's size.

This 'super-critical' violation of the area law for a physical system has inspired follow-up works; most notable are [4, 5] and [1]. A class of *half*-integer spin chains, called Fredkin spin chain [4, 5], was proposed with similar behavior and scaling of the entanglement entropy as in [2].

More recently, a deformation of the Hamiltonian in [2] was proposed by Z. Zhang, A. Ahmadain, I. Klich, in which, the ground state is a superposition of all Motzkin walks weighted by the area between the Motzkin walk and the horizontal axis [1]. The half-chain entanglement entropy of this model violates the area law with the maximum possible scaling factor with the system's size (i.e., n). However, they did not quantify the gap to the first excited state.

In [2] the gap to the first excited state was proved to scale as n^{-c}, where $c \geq 2$. Later it was shown that the gap of Fredkin spin chain has the same scaling with the system's size [8]. Here we prove an upper bound on the gap of the weighted-Motzkin quantum spin-chain proposed in [1] that scales as $8ns\ t^{-n^2/3}$
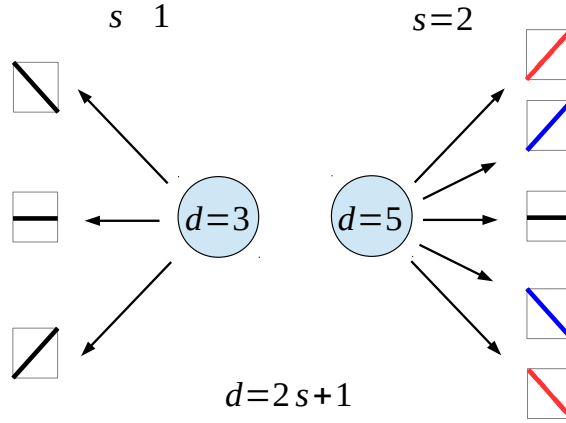
Figure 1: Labels for the $2s+1$ states for $s=1$ and $s=2$. Note that the flat steps are always black for all s.

where $t>1$.

We remind our reader the asymptotic notations:

- $g(n)$ is $O(f(n))$ if and only if for some constants c and n_0, $g(n) \leq cf(n)$ for all $n \geq n_0$,

- $g(n)$ is $\Omega(f(n))$ if for some constants c and n_0, $g(n) \geq cf(n)$ for all $n \geq n_0$,

- $g(n)$ is $\Theta(f(n))$ if $g(n) = O(f(n))$ and $g(n) = \Omega(f(n))$.

Let us denote the gap by Δ. Below if we want to emphasize the gap of a particular Hamiltonian, we write $\Delta(H)$ for the gap. In the table below, we summarize what is known about the recent results that achieve "super-critical" scaling of the entanglement entropy in physical quantum spin-chains:

The model	Spin dimension	Entanglement Entropy Approximately	Gap
[2]	$s > 1$ integer	$\sqrt{n} \log(s)$	$\Theta(n^{-c}) \leq \Delta \leq \Theta(n^{-2}), \quad c \gg 1$
[4, 5]	$s > 1/2$ half-integer	$\sqrt{n} \log(s)$	$\Theta(n^{-c}) \leq \Delta \leq \Theta(n^{-2}), \quad c \gg 1$ [8]
[1]	s positive integer	$n \log(s)$	$\Delta \leq 8ns \, t^{-n^2/3}, \quad t > 1*$
* Proved in this paper.			

II. HAMILTONIAN GAP OF RECENT EXACTLY SOLVABLE MODELS

A. The Motzkin quantum spin chain

The predecessor of [1] is the colored Motzkin spin chain [2], which we now describe. We take the length of the chain to be $2n$ and consider an integer spin$-s$ chain. As before the $d = 2s + 1$ spin states are labeled by up and down steps of s different colors as shown in Fig. 1. Equivalently, and for better readability, we instead use the labels $\{u^1, u^2, \cdots, u^s, 0, d^1, d^2, \cdots, d^s\}$ where u means a step up and d a step down. We distinguish each *type* of step by associating a color from the s colors shown as superscripts on u and d. Lastly, 0 denotes a flat step which always has a single color (black).

A Motzkin walk on $2n$ steps is any walk from $(x, y) = (0, 0)$ to $(x, y) = (2n, 0)$ with steps $(1, 0)$, $(1, 1)$ and $(1, -1)$ that never passes below the $x-$axis, i.e., $y \geq 0$. An example of such a walk is shown in Fig. 2. Each up step in a Motzkin walk has a corresponding down step. When the Motzkin walk is $s-$colored, the up steps are colored arbitrary from $1, \ldots, s$, and each down step has the same color as its corresponding up step (see Fig. 2).

Denote by Ω_j the set of all $s-$colored Motzkin paths of length j. Below for notational convenience we simply write $\Omega \equiv \Omega_{2n}$. In this model the unique ground state is the $s-$colored *Motzkin state* which is defined to be the uniform superposition of all s colorings of Motzkin walks on $2n$ steps

$$|\mathcal{M}_{2n}\rangle = \frac{1}{\sqrt{|\mathcal{M}_{2n}|}} \sum_{x \in \Omega} |x\rangle.$$

The half-chain entanglement entropy is asymptotically [2]

$$S = 2 \log_2 (s) \sqrt{\frac{2\sigma}{\pi}} \sqrt{n} + \frac{1}{2} \log_2 (2\pi\sigma n) + \left(\gamma - \frac{1}{2}\right) \log_2 e \quad \text{bits}$$

where $\sigma = \frac{\sqrt{s}}{2\sqrt{s}+1}$ and γ is Euler's constant. The Motzkin state is a pure state, whose entanglement entropy is zero. However, the entanglement entropy quantifies the amount of disorder produced (i.e., information lost) by ignoring a subset of the chain. The leading order $\sqrt{n}$ scaling of the half-chain entropy establishes that there is a large amount of quantum correlations between the two halves.

Part of the reason that there is $\sqrt{n}$ half-chain entanglement entropy in [2] is that the color of each down step must match the color of its corresponding up step, and order $\sqrt{n}$ of these matched pairs are in opposite halves of the chain. The latter is because, the expected height in the middle of the chain scales as $\sqrt{n}$, which is a consequence of universality of Brownian motion and the convergence of Motzkin walks to Brownian excursions.

Consider the following local operations to any Motzkin walk: interchanging zero with a non-flat step (i.e., $0u^k \leftrightarrow u^k 0$ or $0d^k \leftrightarrow d^k 0$) or interchanging a consecutive pair of zeros with a peak of a given color
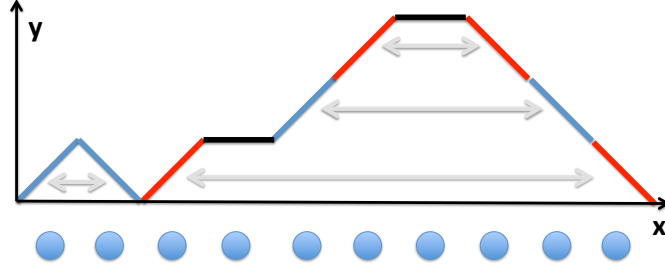
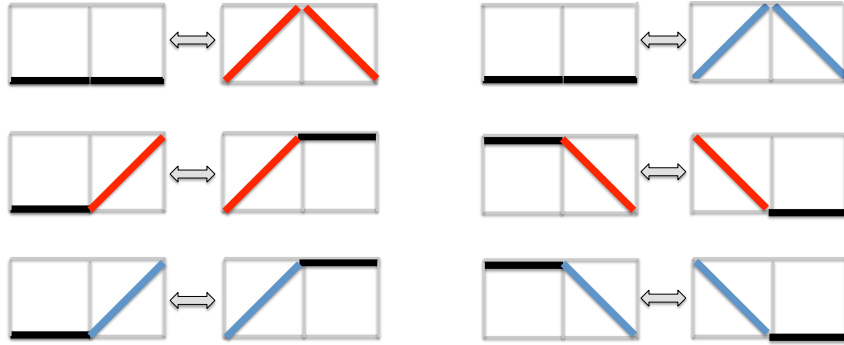Figure 2: A Motzkin walk with $s = 2$ colors on a chain of length $2n = 10$.

Figure 3: Local moves for $s = 2$.

(i.e., $00 \leftrightarrow u^k d^k$). These are shown in Fig. 3. Any $s-$colored Motzkin walk can be obtained from another one by a sequence of these local changes.

To construct a local Hamiltonian with projectors as interactions that has the uniform superposition of the Motzkin walks as its zero energy ground state, each of the local terms of the Hamiltonian has to annihilate states that are symmetric under these interchanges. Local projectors as interactions have the advantage of being robust against certain perturbations [9]. This is important from a practical point of view and experimental realizations.

The local Hamiltonian that has the Motzkin state as its unique zero energy ground state is [2]

$$H = \Pi_{boundary} + \sum_{j=1}^{2n-1} \Pi_{j,j+1} + \sum_{j=1}^{2n-1} \Pi_{j,j+1}^{cross}, \tag{1}$$

where $\Pi_{j,j+1}$ implements the local operations discussed above and is defined by

$$\Pi_{j,j+1} \equiv \sum_{k=1}^{s} \left[|U^k\rangle_{j,j+1}\langle U^k| + |D^k\rangle_{j,j+1}\langle D^k| + |\varphi^k\rangle_{j,j+1}\langle \varphi^k| \right]$$

with $|U^k\rangle = \frac{1}{\sqrt{2}}\left[|0u^k\rangle - |u^k 0\rangle\right]$, $|D^k\rangle = \frac{1}{\sqrt{2}}\left[|0d^k\rangle - |d^k 0\rangle\right]$ and $|\varphi^k\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle - |u^k d^k\rangle\right]$. The projectors $\Pi_{boundary} \equiv \sum_{k=1}^{s}\left[|d^k\rangle_1\langle d^k| + |u^k\rangle_{2n}\langle u^k|\right]$ select out the Motzkin state by excluding all walks that start and end at non-zero heights. Lastly, $\Pi_{j,j+1}^{cross} = \sum_{k\neq i}|u^k d^i\rangle_{j,j+1}\langle u^k d^i|$ ensures that balancing is well ordered (i.e., prohibits $00 \leftrightarrow u^k d^i$); these projectors are required only when $s > 1$ and do not appear in [3].
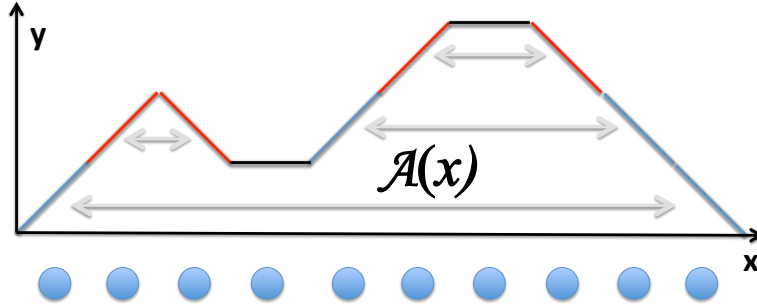
Figure 4: A $2-$colored Motzkin walk, x, with the area $\mathcal{A}(x)$.

B. The weighted Motzkin quantum spin chain

Let $t > 0$ be a real parameter and the state $|GS\rangle$ be defined by

$$|GS\rangle = \frac{1}{\sqrt{Z}} \sum_{x \in \Omega} t^{\mathcal{A}(x)} |x\rangle,$$

where $\mathcal{A}(x)$ is the area enveloped by the Motzkin path and the $x-$axis (See Fig. 4), and $Z \equiv \sum_{x \in \Omega} t^{2\mathcal{A}(x)}$ is the normalization.

Comment: Taking $t = 1$, $|GS\rangle$ becomes the Motzkin state described above. Taking $t < 1$, it was shown in [1] that the ground state will have a half-chain entanglement entropy of $O(1)$.

The local Hamiltonian with the weighted Motzkin state as its unique zero energy ground state is [1]

$$H(t) = \Pi_{boundary} + \sum_{j=1}^{2n-1} \Pi_{j,j+1} + \sum_{j=1}^{2n-1} \Pi_{j,j+1}^{cross}, \tag{2}$$

where $\Pi_{boundary}$ and $\Pi_{j,j+1}^{cross}$ are as in [2]. However, $\Pi_{j,j+1}$ is deformed and depends on a parameter $t > 0$

$$\Pi_{j,j+1}(t) \equiv \sum_{k=1}^{s} \left[|U^k(t)\rangle_{j,j+1} \langle U^k(t)| + |D^k(t)\rangle_{j,j+1} \langle D^k(t)| + |\varphi^k(t)\rangle_{j,j+1} \langle \varphi^k(t)| \right] \tag{3}$$

where $|U^k(t)\rangle = \frac{1}{\sqrt{1+t^2}} \left[t |0u^k\rangle - |u^k 0\rangle \right]$, $|D^k(t)\rangle = \frac{1}{\sqrt{1+t^2}} \left[|0d^k\rangle - t |d^k 0\rangle \right]$ and $|\varphi^k(t)\rangle = \frac{1}{\sqrt{1+t^2}} \left[|u^k d^k\rangle - t |00\rangle \right]$. If $t > 1$, then this deformation favors area-increasing moves that makes the expected height in the middle of the Motzkin walks proportional to n(instead of $\sqrt{n}$). Consequently, the left half of the chain has about order n step ups whose corresponding down steps occur on the right half of the chain. The exponential number of possible colors of the step ups on the left hand side introduces a very large correlation between the two halves of the chain. This results in a highly entangled ground state where the entropy $S = \Theta(n)$.

In [1] they prove that the unique ground state of $H(t)$ is $|GS\rangle$. They found that the half-chain entan-

glement entropy as a function of t and s behaves as

$$S_n = \begin{cases} \Theta(n) & t > 1,\ s > 1 \\ O(1) & t < 1. \end{cases}$$

Note that at $t = 1$ and $s = 1$ the model coincides with [3] and has $S_n = \Theta(\log(n))$, whereas for $t = 1$ and $s > 1$, it coincides with [2] that has $S_n = \Theta(\sqrt{n})$.

The intuition behind the $t-$deformation is that one wants to increase the expected height in the middle of the chain to be $\Theta(n)$. To do so, they multiply the ket in the projector that favors a local increase of the area with a parameter $t > 1$. This would prefer the moves that increase the area and consequently push the whole walk up.

We now prove that the gap of this Hamiltonian for $t > 1$ is $\Delta \leq 8ns\ t^{-n^2/3}$.

C.   Reduction to the gap to that of an underlying Markov chain

There is a general mapping between Stoquastic and frustration free (FF) local Hamiltonians and classical Markov chains with Glauber dynamics [10].

Using Perron-Frobenius theorem, Bravyi and Terhal showed that the ground state $|GS\rangle$ of any stoquastic FF Hamiltonian can be chosen to be a vector with non-negative amplitudes on a standard basis [10], i.e., $x \in \Omega$ such that $\langle x|GS\rangle > 0$.

Fix an integer $s \geq 2$ and a real parameter $t > 1$. An s-colored Motzkin walk of length $2n$ is a sequence $x = (x_1, \ldots, x_{2n})$ with each $x_j \in \{u^1, \ldots, u^s, 0, d^1, \ldots, d^s\}$ starting and ending at height zero with matched colors.

We now restrict the Hamiltonian to the Motzkin subspace, i.e., span of Ω and upper bound the gap in this subspace, which serves as an upper bound on the gap of the full Hamiltonian. Following [10, Sec. 2.1] we define the transition matrix that defines a random walk on Ω

$$P(x, y) = \delta_{x,y} - \beta\sqrt{\frac{\pi(y)}{\pi(x)}}\langle x|H|y\rangle \tag{4}$$

where $\beta > 0$ is real and chosen such that $P(x, y) \geq 0$ and $\pi(x) = \langle x|GS\rangle^2$ is the stationary distribution. The eigenvalue equation $(\mathbb{I} - \beta H)|GS\rangle = |GS\rangle$ implies that $\sum_y P(x, y) = 1$. Therefore, the matrix P really defines a random walk on Ω.

Below we prove that P is a symmetric Markov chain with a unique stationary distribution, which means the eigenvalues of P are real and can be ordered as $1 = \lambda_1 > \lambda_2 \geq \cdots$.

Recall that the spectral gap of a Markov chain is the difference of its two largest eigenvalues, i.e. $1 - \lambda_2(P)$. Also the gap of H is equal to the gap of the matrix

$$M(x,y) = \sqrt{\frac{\pi(y)}{\pi(x)}} \langle x|H|y\rangle$$

because H and M are related by a diagonal similarity transformation. The minus sign in Eq. 4 enables us to obtained the energy gap of the FF Hamiltonian from the spectral gap of the Markov chain P

$$\Delta(H) = \frac{1 - \lambda_2(P)}{\beta}. \tag{5}$$

Let us take $1/\beta = \frac{1}{2ns}\frac{1+t^2}{t^2}$, whereby the entries of the matrix P in Eq. 4 become

$$P(x,y) = \delta_{x,y} - \frac{1}{2ns}\frac{1+t^2}{t^2}\sqrt{\frac{\pi(y)}{\pi(x)}}\langle x|H|y\rangle. \tag{6}$$

Therefore, an upper bound on the gap of the Markov chain provides us with an upper bound on the gap of H via

$$\Delta(H) = \frac{2nst^2}{1+t^2}(1 - \lambda_2(P)). \tag{7}$$

After some preliminaries, we show that P is a transition matrix of a reversible Markov chain with the unique stationary distribution

$$\pi(x) \equiv \frac{t^{2\mathcal{A}(x)}}{Z}, \tag{8}$$

where $Z = \sum_{x\in\Omega_{2n}} t^{2\mathcal{A}(x)}$ as above and obtain an upper bound on $1 - \lambda_2(P)$.

Clearly $\sum_x \pi(x) = 1$ and $0 < \pi(x) \leq 1$. Let us analyze $\langle x|H|y\rangle$, where $x = x_1 x_2 \ldots x_{2n}$ is seen as a string with $x_i \in \{u^1, \ldots, u^s, 0, d^1, \ldots, d^s\}$. Similarly for the string y.

Since $\Pi_{j,j+1}(t)$ acts locally, it should be clear that $\langle x|H|y\rangle = 0$ unless the strings x and y coincide everywhere except from at most two consecutive positions. That is $\langle x|H|y\rangle \neq 0$ if and only if there exists a j such that $\langle x|\Pi_{j,j+1}(t)|y\rangle \neq 0$, which in turn means $x_1 = y_1, x_2 = y_2, \ldots, x_{j-1} = y_{j-1}, x_{j+2} = y_{j+2}, \ldots, x_{2n} = y_{2n}$. Note that here $x_j x_{j+1}$ may or may not be equal to $y_j y_{j+1}$.

In the balanced ground subspace we have

$$\langle x|H|y\rangle = \langle x| \left\{ \sum_{j=1}^{2n-1} \mathbb{I}_{d^{j-1}} \otimes \Pi_{j,j+1}(t) \otimes \mathbb{I}_{d^{2n-j-1}} \right\} |y\rangle = \sum_{j=1}^{2n-1} \langle x_j x_{j+1}|\Pi_{j,j+1}(t)|y_j y_{j+1}\rangle. \tag{9}$$

If $x_j x_{j+1} = y_j y_{j+1}$, we have a 'diagonal term', and if $\langle x|H|y\rangle \neq 0$ and $x_j x_{j+1} \neq y_j y_{j+1}$ we have a 'nonzero off-diagonal term'. The latter occurs if and only if $x_j x_{j+1}$ and $y_j y_{j+1}$ are related by a local move.

Let us look at the largest diagonal term $\langle x|H|x\rangle$, where $x = y = 00\cdots 0$. We have

$$\sum_{j=1}^{2n-1} \langle x_j x_{j+1}|\Pi_{j,j+1}(t)|y_j y_{j+1}\rangle = \sum_{j=1}^{2n-1} \langle x_j x_{j+1}| \left\{ \sum_{k=1}^{s} |\varphi^k(t)\rangle_{j,j+1}\langle\varphi^k(t)| \right\} |x_j x_{j+1}\rangle,$$

because only $|\varphi(t)\rangle$ has a $|00\rangle$ in it. The other projectors vanish. Now Eq. 9 becomes

$$|\varphi^k(t)\rangle_{j,j+1}\langle\varphi^k(t)| = \frac{1}{1+t^2} \left[|u^k d^k\rangle - t\,|00\rangle \right] \left[\langle u^k d^k| - t\,\langle 00| \right]$$

$$= \frac{1}{1+t^2} \left[|u^k d^k\rangle\langle u^k d^k| - t\,|00\rangle\langle u^k d^k| - t|u^k d^k\rangle\langle 00| + t^2\,|00\rangle\langle 00| \right]. \qquad (10)$$

So we have $\langle 00| \left\{ |\varphi^k(t)\rangle_{j,j+1}\langle\varphi^k(t)| \right\} |00\rangle = \frac{t^2}{1+t^2}$ and

$$\langle 00\ldots 0| \sum_{j=1}^{2n-1} \left\{ \sum_{k=1}^{s} |\varphi^k(t)\rangle_{j,j+1}\langle\varphi^k(t)| \right\} |00\ldots 0\rangle = \frac{(2n-1)st^2}{1+t^2}.$$

The smallest non-zero diagonal term in $\langle x|H|x\rangle$ corresponds to a tent shape, where for some $1 \le k \le s$, $x_j = u^k$ for $j < n$ and $x_j = d^k$ for $j > n+1$, and $x_n x_{n+1} = u^k d^k$. In this case, Eq. 9 reads

$$\sum_{j=1}^{2n-1} \langle x| \left\{ \sum_{k=1}^{s} |\varphi^k(t)\rangle_{j,j+1}\langle\varphi^k(t)| \right\} |x\rangle = \langle u_n^k d_{n+1}^k|\varphi^k(t)\rangle_{n,n+1}\langle\varphi^k(t)|u_n^k d_{n+1}^k\rangle = \frac{1}{1+t^2}.$$

Based on this analysis and using the definition of P given by Eq. 6 we conclude

$$0 < P(x,x) \le 1 - \frac{1}{2nst^2}.$$

Next consider the off diagonal terms. Suppose x and y coincide everywhere except at $j, j+1$ position. In this case $\langle x|H|y\rangle \ne 0$ iff $x_j x_{j+1}$ and $y_j y_{j+1}$ are related by a local move. For example suppose $x_j x_{j+1} = 00$ and $y_j y_{j+1} = u^2 d^2$, then the only term that survives in Eq. 10 is $-t\,|00\rangle\langle u^2 d^2|$ in the Hamiltonian. So we have

$$\langle 00| \left\{ |\varphi^k(t)\rangle_{j,j+1}\langle\varphi^k(t)| \right\} |u^2 d^2\rangle = -\frac{t}{1+t^2}\langle 00\,|00\rangle\langle u^2 d^2|u^2 d^2\rangle = -\frac{t}{1+t^2}.$$

Since any local move either increases or decreases the area under the walk by exactly one, in Eq. 8 we have

$$\sqrt{\frac{\pi(y)}{\pi(x)}} = \begin{cases} t & x \to y \text{ by a local move that increases the area} \\ 1/t & x \to y \text{ by a local move that decreases the area} \end{cases}$$

Therefore, in Eq. 6 the nonzero off diagonal terms in which $x \ne y$ are

$$P(x,y) = -\frac{1}{2ns}\frac{1+t^2}{t^2}\sqrt{\frac{\pi(y)}{\pi(x)}}\langle x|H|y\rangle$$

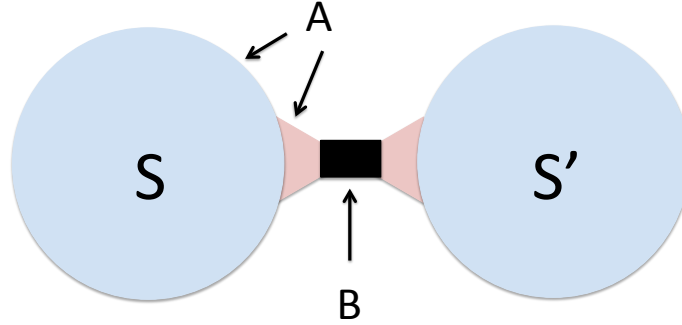Figure 5: The space Ω of colored Motzkin paths contains two large sets S and S' separated by a small bottleneck B. Lemma 2 below verifies that S and S' are large; Lemma 3 that B separates S from S'; and Lemma 4 that B is small. The proof is completed by applying Cheeger's inequality to the set A consisting of all paths reachable by a sequence of local moves starting in S without passing through B.

and the bounds for such terms are

$$0 \leq P(x,y) \leq \frac{1}{2nst^2}.$$

P is a stochastic matrix as the following shows

$$\sum_y P(x,y) = 1 - \frac{1}{(2n-1)s} \frac{1+t^2}{t^2} \frac{1}{\sqrt{\pi(x)}} \langle x|H \sum_y \frac{t^{\mathcal{A}(y)}}{\sqrt{Z}}|y\rangle = 1,$$

where we used $|GS\rangle = \sum_y \frac{t^{\mathcal{A}(y)}}{\sqrt{Z}}|y\rangle$ and the fact that $H|GS\rangle = 0$.

It is easy to see that $P(x,y)$ satisfies detailed balance. Indeed, since $\langle x|H|y\rangle = \langle y|H|x\rangle$, we have $\pi(x)P(x,y) = \pi(y)P(y,x)$. Therefore, $\pi(x)$ is a stationary distribution. Since any two Motzkin walks can be connected by a sequence of local moves [1] P is irreducible, lastly $P(x,x) > 0$, and the chain is aperiodic. We conclude that $\pi(x)$ is the unique stationary distribution that the chain converges to.

D. Upper bound on the gap

For any subset of states $A \subset \Omega$ denote $\pi(A)$ by $\pi(A) \equiv \sum_{x \in A} \pi(x)$. To show that P has a small spectral gap, we are going to identify two large sets (S, S' below) separated by a small bottleneck (B below) and apply Cheeger's inequality (see Eq. 14 below).

Definition. For $a \geq 0$ let $D_a = \{x \in \Omega_{2n} : \mathcal{A}(x) = n^2 - a\}$ be the set of s-colored Motzkin walks of length $2n$ with area defect a whose size we denote by $|D_a|$. Let $p(a)$ be the number of partitions of the integer a.
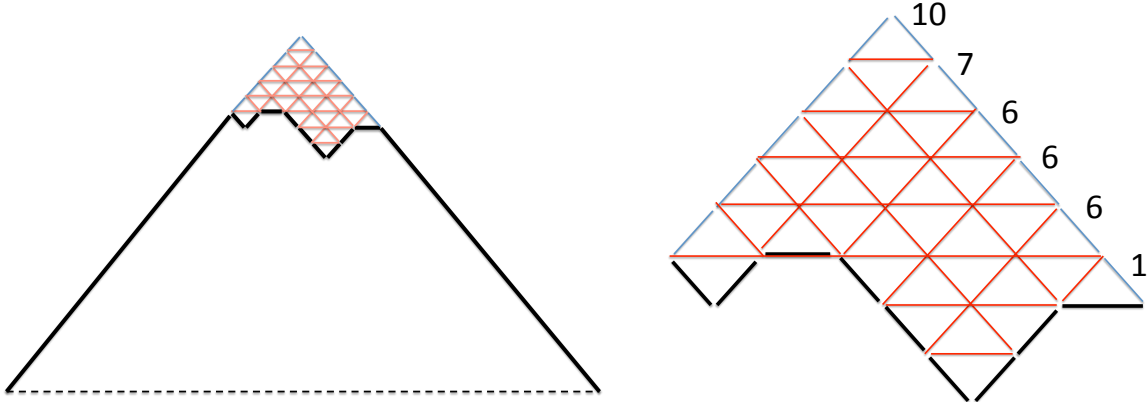
Figure 6: Example of a Motzkin path in D_a with area defect $a = 10 + 7 + 6 + 6 + 6 + 1$. Its complementary partition is zoomed in and shown on the right.

To verify that S and S' are large and that B is small, we will use the following lemma, which says that the stationary distribution π concentrates on walks with small area defect.

Lemma 1. $\pi(D_a) < p(a)t^{-2a} < t^{-a}$

Proof. The shape of a walk $x \in D_a$ is determined by a partition of a, whose triangular diagram lies above x (see Fig. 6). This diagram is analogous to a Young diagram, with triangles instead of squares. Since x has at most n up steps, each of which can be colored arbitrarily in s different ways, we have $|D_a| \leq p(a)s^n$ where $p(a)$ is the number of partitions of a. Using only the s^n tent-shaped paths with full area n^2 to lower bound Z, we obtain.

$$\pi(D_a) = \frac{|D_a|t^{2(n^2-a)}}{Z_{2n}} < \frac{s^n p(a) t^{2(n^2-a)}}{s^n t^{2n^2}} = p(a)t^{-2a}.$$

The asymptotic form of the number of partitions, $p(a)$, is given by Hardy-Ramanujan formula, which is sub-exponential with respect to a [11]:

$$p(a) \sim \frac{1}{4\sqrt{3}a} \exp\left(\pi\sqrt{2a/3}\right).$$

In particular, for fixed $t > 1$ we have for all sufficiently large a,

$$\pi(D_a) = \frac{1}{4\sqrt{3}a} \exp\left(-2a \log t + \pi\sqrt{2a/3}\right) < t^{-a}. \tag{11}$$

$\square$

Definition 1. Let $h_j(x)$ denote the height of x on the jth step. We say x is *prime* if $h_j(x) > 0$ for all $1 < j < 2n$. Denote by Λ the set of all prime colored Motzkin paths in Ω.

In any prime Motzkin walk, the up step in position 1 must have the same color as the down step in position $2n$. Hence $\Lambda = S \cup S'$ where

$$S \equiv \{x \in \Lambda \ : \ color(x_1) = color(x_{2n}) \leq s/2\},$$

$$S' \equiv \{x \in \Lambda \ : \ color(x_1) = color(x_{2n}) > s/2\}.$$

Comment: Possibility of s being odd will not hurt the arguments below.

We now show that nonprime paths have small areas. Let $\Omega_j \times \Omega_{2n-j}$ for the set of concatenations $\tilde{x}\tilde{y}$ where $\tilde{x} \in \Omega_j$ and $\tilde{y} \in \Omega_{2n-j}$. Since $\mathcal{A}(\tilde{x}\tilde{y}) = \mathcal{A}(\tilde{x}) + \mathcal{A}(\tilde{y})$ and $\max_{\tilde{x} \in \Omega_j} \mathcal{A}(\tilde{x}) = j^2/4$, we have

$$\max_{x \in \Omega_j \times \Omega_{2n-j}} \mathcal{A}(x) \leq \frac{j^2 + (2n-j)^2}{4}.$$

Lemma 2. *For any fixed $s \geq 2$ and $t > 1$, we have for all sufficiently large n,*

$$\pi(S') \geq \pi(S) > \frac{1}{4}.$$

Proof. The first inequality follows from the fact that S' includes at least as many colors as S. By symmetry, $\pi\{x \in \Lambda \ : \ color(x_1) = color(x_{2n}) = i\}$ does not depend on $i \in \{1, \ldots, s\}$, and in the worst case where $s = 3$ and $\pi(S)$ is the smallest fraction of $\pi(\Lambda)$ we have

$$\pi(S') \geq \pi(S) \geq \frac{1}{3}\pi(\Lambda)$$

If $x \notin \Lambda$ then x must have a step of height zero in some position $1 < j < 2n$. Then $\mathcal{A}(x) \leq n^2 - 2n + 2$; the bound is saturated when $j = 2$ and we have a concatenation of two tents of heights one and $n-1$. By Eq. 11, we obtain for sufficiently large n

$$\pi(\Omega - \Lambda) \leq \sum_{a \geq 2(n-1)} \pi(D_a) < \sum_{a \geq 2(n-1)} t^{-a} \approx \frac{t^{-2(n-1)}}{1-t}. \tag{12}$$

For sufficiently large n the right side is less than $\frac{1}{4}$, so $\pi(\Lambda) > \frac{3}{4}$ which completes the proof. $\square$

Next we identify the small bottleneck:

$$B \equiv \{\text{colored Motzkin paths with an up step of height zero in position } n\} \cup$$
$$\{\text{colored Motzkin paths with a down step of height zero in position } n+1\}.$$

Lemma 3. *Any sequence of local moves from S to S' must pass through B.*

Proof. Any walk in S starts with a step up of color $i \leq s/2$ and is matched by a down step of color i. For the sake of concreteness call this color i "blue". Any walk in S' starts with an up step and ends with a down step of color $\ell > s/2$, which for the sake of concreteness we call "red". For the walk in S to go to the walk in S' by a sequence of local moves, the up and down blue steps would have to meet and annihilate (i.e. become 00) at some intermediate position. Then a matched pair of red up and down steps would have to be created at some intermediate point $1 < j < 2n$. Eventually the red step up moves all the way to the left (i.e., position 1) via a sequence of local moves and the red step down moves all the way to the right (i.e., position $2n$). If $j \geq n$ then the red up step would necessarily pass through position n; similarly if $j < n$ the red down step would pass through the position $n + 1$. $\qquad\square$

Lemma 4. *For fixed $t > 1$ we have $\pi(B) < t^{-n^2/3}$ for all sufficiently large n.*

Proof. There is a quadratic area cost to being in B: Namely, for any $x \in B$ we have for $n > 1$

$$\mathcal{A}(x) \leq \frac{(n-2)^2 + (n+2)^2}{4} < \frac{1}{2}n^2 + 2n.$$

Hence by Eq. 11 we have for sufficiently large n

$$\pi(B) < \sum_{a \geq \frac{1}{2}n^2 - 2n} \pi(D_a) < \sum_{a \geq \frac{1}{2}n^2 - 2n} t^{-a} < \frac{t^{-\frac{1}{2}n^2 + 2n}}{1 - t} < t^{-n^2/3}. \tag{13}$$

$\qquad\square$

Theorem 1. *For fixed $s \geq 2$ and $t > 1$ we have $1 - \lambda_2(P) < 8nst^{-n^2/3}$, and $\Delta(H) < 8ns\, t^{-n^2/3}$ for all sufficiently large n.*

Proof. We are going to use (the easy part of) Cheeger's inequality for reversible Markov chains, which says that for any $A \subset \Omega$ with $\pi(A) \leq \frac{1}{2}$ we have (See [12, Theorem 13.14 and Eqs (7.4)-(7.6)])

$$1 - \lambda_2 \leq \frac{2Q(A, A^c)}{\pi(A)}, \tag{14}$$

where denoting by $Q(x, y) \equiv \pi(x)P(x, y) = \pi(y)P(y, x)$, we have $Q(A, A^c) \equiv \sum_{x \in A, y \in A^c} Q(x, y)$.

Our choice of A is

$A \equiv \{x \in \Omega \mid$ there exists a sequence of legal moves $x_0, \ldots, x_k = x$ with $x_0 \in S$ and all $x_i \notin B\}$.

By definition, every edge from A to A^c is an edge from A to B, so by Lemma 4 and sufficiently large n we have

$$Q(A, A^c) = \sum_{x \in A, y \in B} Q(x, y) \leq \sum_{x \in \Omega, y \in B} \pi(y)P(y, x) = \pi(B) < t^{-n^2/3}$$

On the other hand, by Lemma 2

$$\pi(A) \geq \pi(S) > \frac{1}{4}.$$

Finally, we need to show that $\pi(A) \leq \frac{1}{2}$ so that Cheeger's inequality may be applied. To do so, we show that if $x \in A$ then $x' \in A^c$ for any x' that is a Motzkin walk of the same shape as x whose colors at each step are changed according to $i \to s - i + 1$. Indeed, since $x \in A$ there is a sequence of legal moves $x_0, \ldots, x_k = x$ with $x_0 \in S$ and all $x_i \notin B$. Noting that $B = B'$, we have that $x', x'_{k-1}, \ldots, x'_0$ is a sequence of local moves from x' to $x'_0 \in S'$ with all $x'_i \notin B$. By Lemma 3 it follows that any sequence of local moves from S to x' must pass through B. Hence $x' \notin A$.

By Eq. 14 we conclude that $1 - \lambda_2 < 8t^{-n^2/3}$. This is the desired result because from Eq. 7 we have ($t > 1$)

$$\Delta(H) = \frac{2nst^2}{1+t^2}(1 - \lambda_2(P)) < 8ns\, t^{-n^2/3}. \tag{15}$$

$\square$

This shows that so far the only model that violates the area-law by a factor of n in one-dimension and satisfies physical reasonability criteria of translational invariance, locality and uniqueness of ground state, has an exponentially small energy gap. It would be very interesting if a physically reasonable model could be proposed in which the half-chain entanglement entropy violates the area law by n and that the gap would vanish as a power-law in n.

III.   ACKNOWLEDGEMENTS

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* Electronic address: levine@math.cornell.edu

† Electronic address: q.eigenman@gmail.com

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