Hall's Marriage Theorem and Hamiltonian Cycles in Graphs

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May, 2001

If S is a set of vertices in a graph G, let $d(S)$ be the number of vertices in G adjacent to at least one member of S. The following result is known as Phillip Hall's marriage theorem.

Proposition 1 (Hall, 1935). Suppose A, B, $|A| = |B| = n$ are the parts of a bipartite graph with the property that $|d(S)| \geq |S|$ for every $S \subset A$. Then there exists a bijection $f : A \to B$ such that a is adjacent to $f(a)$ for all $a \in A$.

Proof. Induct on n. Suppose first that there exists a proper subset $S \subset A$ such that $|d(S)| = |S|$. Then for any $T \subset A \backslash S$ we have

$$|d(T) \cap (B \backslash d(S))| \geq |d(S \cup T)| - |d(S)| \geq |T|,$$

so the induced bipartite graph on $A \backslash S$, $B \backslash d(S)$ also satisfies the hypothesis of the proposition. Inductively, we obtain bijections $S \to d(S)$ and $A \backslash S \to B \backslash d(S)$.

If, on the other hand, there is no proper $S \subset A$ such that $|d(S)| = |S|$, choose any adjacent pair $a \in A$ and $b \in B$. Given any $T \subset A \backslash \{a\}$, we have

$$|d(T) \cap (B \backslash \{b\})| \geq |d(T)| - 1 \geq |T|.$$

Inductively, we obtain a bijection $f : A \backslash \{a\} \to B \backslash \{b\}$. Extend f by putting $f(a) = b$.

If we identify the sets A and B in Hall's theorem, we obtain a result known as Tutte's 2-matching theorem. A graph G is said to have a 2-matching if its vertex set can be paritioned into parts of size ≥ 2 such that the induced subgraph on each part has a Hamiltonian cycle (here the graph consisting of two vertices joined by an edge is considered to have a Hamiltonian cycle). The following result characterizes graphs that have a 2-matching.

Proposition 2 (Tutte, 1953). A graph G has a 2-matching if and only if every coclique C of G satisfies $|d(C)| \geq |C|$.

Proof. If G has a 2-matching, then to verify the $|d(C)| \geq |C|$ condition for G, we need only verify it for the graph G' consisting of the cycles in the 2-matching. Since these cycles are disjoint, this further reduces to the case when G' consists of a single cycle. Now given a coclique C in G', each point $x \in C$ is adjacent to two points not in C, while each $y \in G' \backslash C$ is adjacent to at most two points of C, so $|d(C)| \geq |C|$.

For the converse, consider the bipartite graph on two disjoint copies of the vertex set of G, with x in one copy adjacent to y in the other whenever x and y are adjacent in G. Given any set S of vertices of G, let C be the set of points isolated in S. Then $d(C)$ and $S \backslash C$ are disjoint and both contained in $d(S)$, so

$$|d(S)| \geq |d(C)| + |S \backslash C| \geq |S|,$$

in other words, G' satisfies the hypothesis of Hall's theorem. We obtain a permutation σ of the vertices of G such that x is adjacent to $\sigma(x)$ for all x, and the cycle structure of σ yields a 2-matching.

It would be of interest to strengthen the hypothesis of Hall's theorem in such a way that σ can be taken cyclic, and thus obtain a Hall-type condition that is sufficient to ensure that a graph has a Hamiltonian cycle. The following result shows that the most straightforward generalization of Hall's condition is not sufficient.

Proposition 3. For every integer $r \geq 0$, there exists a graph G on $n > r$ vertices such that

(i) For any set S of vertices of G, if $1 \leq |S| \leq n - r$ then $|d(S)| \geq r + |S|$;
(ii) G is not Hamiltonian.

Proof. We give a general construction of such graphs for $r \geq 2$. Of

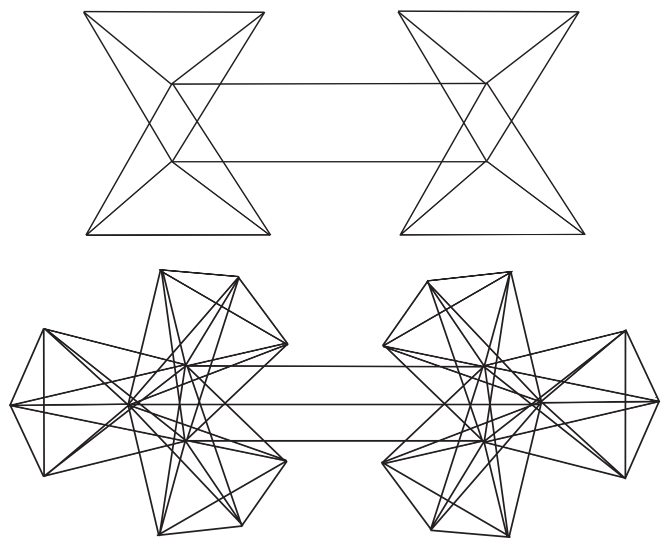

Figure 1: The construction of G in the cases $r = 2$ and $r = 3$

course, this implies the cases $r = 0$ and $r = 1$ as well. Let G consist of two "wheels" W and W', each composed of a "hub" H, H' and a "rim" R, R'. The hubs consist of r vertices each, $H = \{h_1, \ldots, h_r\}$, $H' = \{h'_1, \ldots, h'_r\}$ and they are connected by "axles:" edges (h_k, h'_k), $k = 1, \ldots, r$. The rims consist of r^2 vertices each, divided into r complete graphs on r vertices each. We index them $R = \{r_{ij}\}$, $R' = \{r'_{ij}\}$, $i, j = 1, \ldots, r$. For any i and any $j_1 \neq j_2$ there are edges (r_{ij_1}, r_{ij_2}) and (r'_{ij_1}, r'_{ij_2}). So each $R_i = \{r_{ij}\}_{j=1,\ldots,r}$ and $R'_i = \{r'_{ij}\}_{j=1,\ldots,r}$ is a complete graph on r vertices. Finally, we add the "spokes:" for every i, j and k, there are edges (r_{ij}, h_k) and (r'_{ij}, h'_k).

We show first that G is not Hamiltonian. The $2r$ rim sectors $R_1, \ldots, R_r$ and $R'_1 \ldots R'_r$ are connected to each other only via the hubs H and H'. Thus whenever a path crosses from one sector to another, it must pass through one of the hubs. Furthermore, if a path crosses between sectors on opposite wheels, say from R_i to R'_j, then it must pass through two hub points, one on each wheel. In order to visit all $2r$ sectors, a Hamiltonian cycle would have to cross between wheels at least twice, and would have to switch sectors at least $2r$ times, so it would have to pass through at least $2r + 2$ hub points. This is impossible, because the total number of hub points is only $2r$.

It remains to show that G satisfies property (i). Suppose first that S is a set of vertices in wheel W. If $S \subset H$, then S borders all of R as well as $|S|$ elements of H', for a total of

$$|d(S)| = r^2 + |S| \geq r + |S|.$$

If S intersects both R and H, say it contains k vertices in R and l vertices of H, then S borders all of W as well as l elements of H', which gives

$$|d(S)| = r^2 + r + l \geq k + r + l = r + |S|.$$

Finally, suppose $S \subset R$. If S intersects the sector R_i, then it borders either all but one vertex of R_i (if $|S \cap R_i| = 1$) or all of R_i (if $|S \cap R_i| > 1$). Since $r \geq 2$, this implies that $|d(S) \cap R_i| \geq |S \cap R_i|$, and so $|d(S) \cap R| \geq |S|$. Since S also borders all of H, we get $|d(S)| \geq r + |S|$.

This establishes (i) in the case when $S \subset W$, and, by symmetry, when $S \subset W'$. Suppose now that S is any set of vertices of G, and let $T = S \cap W$, $T' = S \cap W'$. If $|d(T) \cap d(T')| \leq r$, then

$$\begin{aligned} |d(S)| &= |d(T)| + |d(T')| - |d(T) \cap d(T')| \\ &\geq (r + |T|) + (r + |T'|) - r = r + |S|. \end{aligned}$$

On the other hand, if $|d(T) \cap d(T')| > r$, then $|S \cap (H \cup H')| > r$, so S intersects both H and H'. Then S borders all of R and R', so $|d(S)| \geq 2r^2 + r + 1$. So there's no problem unless $|S| > 2r^2 + 1 > r^2 + 2r + 1$, in which case $|S|$ must intersect both R and R', and then $d(S) = G$. This completes the proof.

At this point, a natural question arises: Does there exist a function f such that any graph G satisfying

$$|d(S)| \geq \min(f(|S|), |V(G)|)$$

for all $S \subset V(G)$ is Hamiltonian? We show that $f(k) = 2k$ is sufficient. This follows easily from a well-known result of Dirac.

Proposition 4 (Dirac). If G is a graph on n vertices and every vertex of G has degree at least $n/2$. then G has a Hamiltonian cycle.

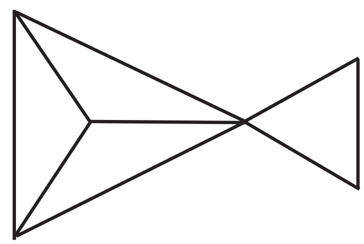

Figure 2: A non-Hamiltonian graph on 6 vertices satisfying $|d(S)| \geq \min(4|S|/3, 6)$ for all $|S|$.

Proof. Induct on the number of pairs of non-adjacent vertices of G. Given non-adjacent vertices x and y in G, by the inductive hypothesis there is a Hamiltonian path in G from x to y, call it $x = v_1, v_2 \ldots, v_n = y$. Because x and y have degree at least $n/2$, there exists an index i such that y is adjacent to v_i and x is adjacent to v_{i+1}. Then $x, v_2, v_3, \ldots, v_i, y, v_{n-1}, v_{n-2}, \ldots, v_{i+1}, x$ is a Hamiltonian cycle.

Now suppose G is a graph on n vertices with the property that $|d(S)| \geq \min(2|S|, n)$ for all sets S of vertices of G. In particular, any $\lceil n/2 \rceil$ vertices border all of G, so every vertex of G can miss at most $\lfloor n/2 \rfloor$ other vertices, i.e. every vertex has degree at least $\lceil n/2 \rceil$, so Dirac's result applies.

The fact that we used the condition $|d(S)| \geq min(2|S|, n)$ only for S of cardinality $\lceil n/2 \rceil$ indicates that it may be possible to replace 2 with some smaller constant. It would be of interest to compute

$$\lambda = \inf\{c \leq 2 : \text{every graph } G \text{ satisfying } |d(S)| \geq min(c|S|, n) \text{ for all } |S| \text{ is Hamiltonian}\}.$$

The graph in Figure 2 shows that $\lambda > 4/3$.

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